Groupe de Recherche en
Économie Théorique et Appliquée

Solving the Hotelling Model

in Feedback Form

Sébastien ROUILLON

Université de Bordeaux
GREThA UMR CNRS 5113

Cahiers du GREThA
n° 2011-19

GRETHA UMR CNRS 5113
Université Montesquieu Bordeaux IV
Avenue Léon Duguit - 33608 PESSAC - FRANCE
Tel : +33 (0)5.56.84.25.75 - Fax : +33 (0)5.56.84.86.47 - www.gretha.fr

Une solution du modèle d'Hotelling sous la forme d'une règle de décision en boucle fermée

Résumé

Nous caractérisons la trajectoire optimale d'extraction d'une ressource épuisable, sous la forme d'une règle de décision en boucle bouclée, applicable pour une large classe de modèles.

Mots-clés : Règle d'Hotelling; ressource épuisable; programmation dynamique.

Solving the Hotelling Model in Feedback Form

Abstract

We give a characterization, in feedback form, of the optimal extraction path of an exhaustible resource, which holds for a large class of models.

Keywords: Hotelling rule; exhaustible resource; dynamic programming.

JEL : Q30, C61

Reference to this paper: ROUILLON Sébastien (2011) Solving the Hotelling Model in Feedback Form, *Cahiers du GREThA*, n°2011-19.

http://ideas.repec.org/p/grt/wpegrt/2011-19.html.

GRETHA UMR CNRS 5113
Université Montesquieu Bordeaux IV
Avenue Léon Duguit - 33608 PESSAC - FRANCE
Tel : +33 (0)5.56.84.25.75 - Fax : +33 (0)5.56.84.86.47 - www.gretha.fr

1 Introduction

Hotelling's seminal paper (1931) is the cornerstone of the economics of exhaustible resources. He established the now-famous "Hotelling rule," which states that the marginal utility of extracting/consuming the exhaustible resource must grow at a rate equal to the rate of interest, along an optimal extraction path. Combined to the prescription that the depletion of the resource is optimal, in finite or infinite time (depending on the limit of the marginal utility when consumption tends to zero), the Hotelling rule allows to derive the efficient extraction path, in two steps. One must first determine the optimal time of depletion, using appropriate transversality conditions. And one must then find the initial rate of extraction, such that the corresponding extraction path (induced by the Hotelling rule) will effectively deplete the resource at the optimal time.

In this paper, we propose an alternative way to derive an optimal extraction path of an exhaustible resource. Formally, we display a characterization, *in feedback form*, of the optimal solution. We believe that this rule will prove easier to use in many situations. Moreover, it holds for a large class of exhaustible resource problems, as long as the resource stock is not an argument of the objective function.

The remaining of the paper is organized as follows. Section 2 recalls the basic Hotelling model. Section 3 states the optimal rule in feedback form and his proof.

2 The Basic Hotelling Model.

The economy is endowed with a finite stock, x_0, of a homogeneous consumption good. Denote by q the rate of consumption and by $u(q)$ the instantaneous utility of consuming q. We assume that there exists $\overline{q} > 0$ (including the possibility that $\overline{q} = \infty$) such that $u'(q) > 0$ and $u''(q) < 0$, for all $0 \leq q < \overline{q}$, and $u(q) = u(\overline{q})$, for all $q \geq \overline{q}$. Social welfare is taken to be the sum of utilities, discounted at the strictly positive rate δ. The social problem is therefore to find a consumption path, $q(\cdot)$, that maximizes

$$\begin{array}{l} W = \int_0^\infty u\left(q\left(t\right)\right) e^{-\delta t} dt \\ \text{subject to:} \\ \dot{x}\left(t\right) = -q\left(t\right),\, x\left(0\right) = x_0, \\ q\left(t\right) \geq 0,\, x\left(t\right) \geq 0. \end{array} \tag{1}$$

This problem is usually solved by means of the Pontryagin's maximum principle. This leads to the characterization of an optimal solution as a consumption path satisfying the Hotelling rule (i.e., such that the marginal utility increases at the rate of discount) and depleting the resource stock in finite or infinite time (depending on the limit of the marginal utility for an arbitrarily small consumption).

3 Optimal Solution in Feedback Form.

This section states and discusses the result of the paper, taking the form of proposition 1 below. We actually solve the Hotelling problem by means of dynamic programming. This allows a characterization in feedback form of the optimal solution. The proofs are relegated in the appendix.

In order to write Proposition 1, further definitions and assumptions are needed. Define $\sigma(q) \equiv -u''(q)\, q/u'(q)$, for all $0 \leq q < \overline{q}$, the elasticity of marginal utility, and $S(q) \equiv u(q) - u'(q)\, q$, for all $0 \leq q \leq \overline{q}$, the consumer's surplus. Assuming integrability of $\sigma(q)$, let $\Theta(q) \equiv \int_0^q \sigma(s)\, ds$, for all $0 \leq q < \overline{q}$, and assume that $\lim_{q \to \overline{q}} \Theta(q) = \infty$.

Proposition 1. (Optimal policy in Feedback form.)
The optimal policy, given in the feeback form $q = f(x)$, is implicitly defined by

$$\Theta(f(x)) = \delta x, \text{ for all } x. \tag{2}$$

The corresponding value function is

$$V(x) = (1/\delta)\, S(f(x)), \text{ for all } x. \tag{3}$$

The figure below illustrates the utilization of proposition 1. It displays a possible graph of the elasticity of the marginal utility $\sigma(q)$. According to Proposition 1, it is optimal, at any time, to extract a quantity q°, such that the surface area below the graph, from the origin to q°, equals δx_0, where δ is the discount rate and x_0 is the current state of the resource.

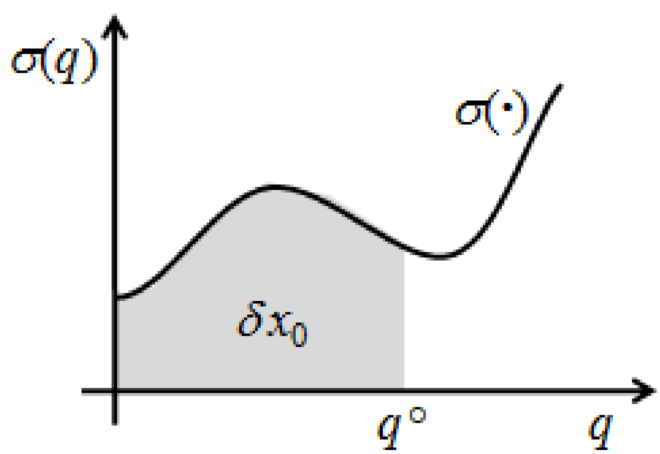

Figure 1. Derivation of the optimal extraction.

4 Discussion

Following from Proposition 1 and Figure 1, the solution to the Hotelling problem only depends on the curvature of the marginal utility, $\sigma(\cdot)$, the rate of discount, δ, and the initial stock of the resource, x_0. Here, we derive, by means of comparative statics, several properties that follows from this.

To begin with, let us analyse the role of the elasticity of the marginal utility. Therefore, we consider two Hotelling problems, only differing with respect to their elasticity of marginal utility, $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ (say). We can use Figure 1 to deal with the comparative statics. We know that the optimal extraction is obtained when the area below the graph of the elasticity of the marginal utility has measure δx_0. From this, if we assume that $\sigma_1(q) < \sigma_2(q)$, for all q, for example, it will be optimal to extract more of the resource in the first problem, than in the second one. Hence, this shows that *the larger the elasticity of marginal utility is, the smaller the optimal extraction.*

Let us now view the role played by the rate of discount and the resource stock. We can consider these two parameters jointly, since the optimal extraction actually appears to be a function of their product δx_0 only. To the best of my knowledge, this stricking property is never mentionned (see, in particular, Dasgupta and Heal, 1979). Using Figure 1, one immediately sees that *the larger the rate of discount and/or the resource stock are, the larger the optimal extraction.*

It is of interest to clarify the relationship existing between Proposition 1 and the Hotelling rule. Of course, the policy in feedback form (2), being optimal, induces an extraction path that satisfies the Hotelling rule. This implication is shown in the appendix (see Lemma 3). However, the reciprocal implication is false. Indeed, the Hotelling rule does determine an optimal extraction path, only when combined with an appropriate boundary condition, called a transversality condition. Hence, we can say that *the policy in feedback form (2) summarizes both the Hotelling rule and the transversality condition.*

A last issue of interest concerns the way to obtain similar results in other contexts. The following steps, which actually sketches the proof of proposition 1, provides the trick. It may be fruitful in differential games applied to non-renewable resources (Dockner et al., 2000).

Note that Problem (1) is an autonomous optimal control problem with infinite horizon. Hence, both the value function, $V(x)$, and the optimal control, $f(x)$, are functions of the state alone. They satisfy the Hamilton-Jacobi-Bellman equation

$$\begin{aligned} \delta V(x) &= \max_{q \geq 0} \{u(q) - V'(x)\, q\}, \qquad (4) \\ &= u(f(x)) - V'(x)\, f(x). \end{aligned}$$

As a maximizer of the RHS of (4), $f(x)$ satisfies the complementarity slackness conditions

$$u'(f(x)) - V'(x) \leq 0, \; f(x) \geq 0 \text{ and } (u'(f(x)) - V'(x)) f(x) = 0. \tag{5}$$

Hence, after substitution (using $V'(x) f(x) = u'(f(x)) f(x)$), we can write

$$\delta V(x) = u(f(x)) - u'(f(x)) f(x).$$

By differentiation, we get

$$\delta V'(x) = u''(f(x)) f(x) f'(x).$$

Now, it is natural to assume that $f(x) > 0$, for all $x > 0$. Then, we can subsitute $V'(x) = u'(f(x))$ (from (5)) to obtain the first-order differential equation

$$-\frac{u''(f(x)) f(x)}{u'(f(x))} f'(x) = \sigma(f(x)) f'(x) = \delta$$

from which the feedback form in Proposition 1 derives by integration, using the boundary condition $f(0) = 0$.

5 Aknowlegements.

I wish to thank Robert Cairns and Marc Leandri for their comments, which helped me to improve this paper.

6 Appendix.

Appendix A.1. Proof of proposition 1.

Recall the definitions $S(q) \equiv u(q) - u'(q) q$, for all $0 \leq q \leq \overline{q}$, and $\Theta(q) \equiv \int_0^q \sigma(s) \, ds$, for all $0 \leq q < \overline{q}$, and the assumption that $\lim_{q \to \overline{q}} \Theta(q) = \infty$.

Consider the policy $f(x)$ such that $\Theta(f(x)) = \delta x$, for all x.

1) Feasibility. We first show, in Lemmas 1 and 2, that the policy $f(x)$ is feasible.

Lemma 1. For all $x > 0$, $f(x) > 0$, and $f(0) = 0$.

Proof. If $x = 0$, $f(0) = 0$ follows from $\Theta(0) = 0$. Likewise, $\lim_{x \to \infty} f(x) = \overline{q}$ follows from $\lim_{q \to \overline{q}} \Theta(q) = \infty$. If $0 < x < \infty$, as $\Theta(0) = 0 < \delta x < \infty = \lim_{q \to \overline{q}} \Theta(q)$ and $\Theta(q)$ is continuous, there exists $0 < f(x) < \overline{q}$ such that $\Theta(f(x)) = \delta x$ (by the intermediate value theorem). As $\Theta(q)$ is increasing (for $\Theta'(q) = \sigma(q) > 0$, for all $q > 0$), this solution is unique. $\square$

Lemma 2. The policy $f(x)$ generates a trajectory of the resource stock such that $x(t) \geq 0$, for all t, and $\lim_{t\to\infty} x(t) = 0$.

Proof. By definition, the policy $f(x)$ generates a trajectory such that $\dot{x}(t) = -f(x(t))$, for all t, with initial condition $x(0) = x_0$. Lemma 2 follows directly from the properties of $f(x)$, which implies that $\dot{x}(t) = -f(x(t)) < 0$, when $x(t) > 0$, and $\dot{x}(t) = 0$, when $x(t) = 0$. □

2) Hotelling rule. Assume that the (feasible) policy $f(x)$ is implemented. Denote by $x(\cdot)$ the resulting trajectory of the resource (such that $\dot{x}(t) = -f(x(t))$, for all t, and $x(0) = x_0$). Let T represent the time of depletion of the resource stock (including the possibility that $T = \infty$). Lemma 3 shows the relation between the policy $f(x)$ and the Hotelling rule.

Lemma 3. The policy $f(x)$ satisfies $\sigma(f(x)) f'(x) = \delta > 0$, for all $x > 0$. It follows that, along the path $x(\cdot)$ induced by $f(x)$, the marginal utility $u'(f(x(t)))$ grows at the rate δ, for all $t < T$ (i.e., the Hotelling rule).

Proof. The first part is obtained from the implicit function theorem, which states that $\sigma(f(x)) f'(x) = \delta > 0$, for all $x > 0$.

To show the second part, define, for all $t < T$, $p(t) = u'(f(x(t)))$. Since $x(t) > 0$ is true, for all $t < T$, differentiation yields:

$$\dot{p}(t) = u''(f(x(t))) f'(x(t)) \dot{x}(t).$$

Substituting $\dot{x}(t) = -f(x(t))$ and dividing by $p(t) = u'(f(x(t)))$, we get:

$$\frac{\dot{p}(t)}{p(t)} = \sigma(f(x(t))) f'(x(t)) = \delta,$$

which yields the second part of the lemma. □

3) Objective Value. We derive the value of the social objective (1) when the (feasible) policy $f(x)$ is implemented, through Lemmas 4 and 5.

By definition, the value of the objective function associated with the policy $f(x)$ is:

$$W = \int_0^\infty u(f(x(t))) e^{-\delta t} dt$$
$$\text{where: } \dot{x}(t) = -f(x(t)) \text{ and } x(0) = x_0.$$

Lemma 4. For all $t < T$, $S(f(x(t))) \left(-e^{-\delta t}/\delta\right)$ is a primitive of $u(f(x(t))) e^{-\delta t}$.

Proof. For all $t < T$, let $\phi(t) = S(f(x(t))) \left(-e^{-\delta t}/\delta\right)$. Differentiation yields:

$$\dot{\phi}(t) = S(f(x(t))) e^{-\delta t} + S'(f(x(t))) f'(x(t)) \dot{x}(t) \left(-e^{-\delta t}/\delta\right).$$

After substitution, we get (using $S(q) = u(q) - u'(q)q$, $S'(q) = -u''(q)q$, $\sigma(q) \equiv -u''(q)q/u'(q)$ and $\dot{x}(t) = -f(x(t))$, and rearranging):

$$\dot{\phi}(t) = u(f(x(t)))e^{-\delta t} - (\sigma(f(x(t)))f'(x(t)) - \delta)u'(f(x(t)))f(x(t))\left(-e^{-\delta t}/\delta\right). \tag{6}$$

By Lemma 3, the second term is nil and we get:

$$\dot{\phi}(t) = u(f(x(t)))e^{-\delta t},$$

which is Lemma 4. □

Lemma 5. The value of the social objective W associated to the policy $f(x)$ is $(1/\delta)S(f(x_0))$.

Proof. We treat separately the cases where the resource is depleted asymptotically (i) and in finite time (ii).

(i) Consider the case where $T = \infty$ (i.e., $x(t) > 0$ and $f(x(t)) > 0$, for all t).

By definition of W and Lemma 4, we have (using $x(0) = x_0$):

$$\begin{aligned} W &= \lim_{T\to\infty}\int_0^T u(f(x(t)))e^{-\delta t}dt \\ &= \lim_{T\to\infty}\left[S(f(x(t)))\left(-e^{-\delta t}/\delta\right)\right]_0^T \\ &= (1/\delta)S(f(x_0)) + \lim_{T\to\infty} S(f(x(T)))\left(-e^{-\delta T}/\delta\right) \end{aligned}$$

Therefore, Lemma 5 holds if $\lim_{T\to\infty} S(f(x(T)))\left(-e^{-\delta T}/\delta\right) = 0$.

Remember $\phi(t) = S(f(x(t)))\left(-e^{-\delta t}/\delta\right)$, for all t. Using Lemma 4, we have (using $\dot{\phi}(t) = u(f(x(t)))e^{-\delta t}$, $\phi(t) = S(f(x(t)))\left(-e^{-\delta t}/\delta\right)$ and $S(q) = u(q) - u'(q)q$):

$$\dot{\phi}(t) + \delta\phi(t) = f(x(t))u'(f(x(t)))e^{-\delta t}.$$

From Lemma 3, $u'(f(x(t)))e^{-\delta t} = u'(f(x_0))$, for all t (using $x(0) = x_0$). Substituting, we obtain:

$$\dot{\phi}(t) + \delta\phi(t) = f(x(t))u'(f(x_0)).$$

Since $\lim_{t\to\infty} f(x(t)) = 0$, a solution of this ODE must be such that $\lim_{t\to\infty}\phi(t) = 0$ (see Appendix A.2). This proves that $\lim_{T\to\infty} S(f(x(T)))\left(-e^{-\delta T}/\delta\right) = 0$.

(ii) Consider the case where $T < \infty$ (i.e., $x(t) > 0$ and $f(x(t)) > 0$, for all $t < T$; and $x(t) = 0$ and $f(x(t)) = 0$, for all $t \geq T$).

By definition of W and lemma 4, we have:

$$\begin{aligned} W &= \int_0^T u(f(x(t)))e^{-\delta t}dt + \int_T^\infty u(0)e^{-\delta t}dt \\ &= \left[S(f(x(t)))\left(-e^{-\delta t}/\delta\right)\right]_0^T + u(0)\left(e^{-\delta T}/\delta\right) \\ &= (1/\delta)S(f(x_0)) + (S(f(x(T))) - u(0))\left(-e^{-\delta T}/\delta\right) \end{aligned}$$

Therefore, Lemma 5 holds if $(S(f(x(T))) - u(0))\left(-e^{-\delta T}/\delta\right) = 0$, which is immediate (using $x(T) = 0$, by definition of T, $f(0) = 0$ and $S(q) = u(q) - u'(q)q$). $\square$

4) Optimality. We finally show that the policy $f(x)$ is optimal, by proving that the corresponding value of the objective function, $W = (1/\delta)\, S(f(x))$, satisfies the Hamilton-Jacobi-Bellman equation, for any initial state x, with $q = f(x)$ the optimal control.

Lemma 6. Define $V(x) = (1/\delta)\, S(f(x))$, for all $x > 0$. It satisfies the Hamilton-Jacobi-Bellman equation: $\delta V(x) = \max_{q \geq 0}\{u(q) - V'(x)q\}$. The right-hand side of the Hamilton-Jacobi-Bellman equation is maximized when $q = f(x)$.

Proof. For all $x > 0$, let $V(x) = (1/\delta)\, S(f(x))$. By differentiation, we obtain (using $S'(q) = -u''(q)q$ and $\sigma(q) \equiv -u''(q)q/u'(q)$):

$$V'(x) = \frac{1}{\delta}\sigma(f(x))\, f'(x)\, u'(f(x)).$$

Using lemma 3, it follows that:

$$V'(x) = u'(f(x)).$$

For all $x > 0$, let:

$$H^0(x) = \max_{q \geq 0}\{u(q) - u'(f(x))\, q\}.$$

This function is maximized if, and only if (by concavity of $u(q)$):

$$u'(q) - u'(f(x)) \leq 0,\ q \geq 0 \text{ and } (u'(q) - u'(f(x)))\, q = 0.$$

Hence, $q = f(x)$ maximizes $H^0(x)$ and (using $S(q) = u(q) - u'(q)q$):

$$H^0(x) = S(f(x)).$$

This proves that $V(x) = (1/\delta)\, S(f(x))$ satisfies the Hamilton-Jacobi-Bellman equation: $\delta V(x) = \max_{q \geq 0}\{u(q) - V'(x)q\}$. $\square$

This completes our proof of proposition 1. $\blacksquare$

Appendix A.2.

Let $\phi_0 = (1/\delta)\, S(f(x_0))$ and, for all t, $a(t) = f(x(t))\, u'(f(x_0))$. Knowing that $a(t)$ is positive, decreasing and tends to zero when t tends to infinity, we must show that the solution $\phi(t)$ of the following ODE:

$$\dot{\phi}(t) + \delta\phi(t) = a(t),\ \phi(0) = \phi_0$$

is such that $\lim_{t\to\infty} \phi(t) = 0$.

Indeed, the solution of this ODE is:

$$\phi(t) = \phi_0 e^{-\delta t} + \int_0^t a(s)\, e^{\delta(s-t)} ds.$$

As $a(t)$ is decreasing, $a(t) \leq a(0)$, for all t. As $\lim_{t\to\infty} a(t) = 0$, for all $\varepsilon > 0$, there exists t_0 such that $a(t) < \delta\varepsilon/2$, for all $t > t_0$. Thus, for all $t > t_0$, we have:

$$\begin{aligned} |\phi(t)| &< |\phi_0|\, e^{-\delta t} + a(0) \int_0^{t_0} e^{\delta(s-t)} ds + \delta\varepsilon/2 \int_{t_0}^t e^{\delta(s-t)} ds \\ &< |\phi_0|\, e^{-\delta t} + (a(0)/\delta)\left(e^{\delta t_0} - 1\right) e^{-\delta t} + (\varepsilon/2)\left(1 - e^{\delta(t_0 - t)}\right) \end{aligned}$$

The RHS tends to $\varepsilon/2$ when t tends to infinity. Therefore, $|\phi(t)|$ is smaller than ε if t is sufficiently large. As ε is any small positive number, this proves that $\lim_{t\to\infty} \phi(t) = 0$.

7 References.

Dasgupta, P.S. and G. Heal, "Economic Theory and Exhaustible Resources", Cambridge University Press, 1979.

Dockner, E. et al. (2000), Differential games in economics and management science, Cambridge university press, New York.

Hotelling, H. (1931), "The Economics of Exhaustible Resources", Journal of Political Economy, 39(2): 137-175.

Stiglitz, J.E. (1976), "Monopoly and the Rate of Extraction of Exhaustible Resources", American Economic Review, 66(4): 655-661.

Cahiers du GREThA
Working papers of GREThA

GREThA UMR CNRS 5113

Université Montesquieu Bordeaux IV
Avenue Léon Duguit
33608 PESSAC - FRANCE
Tel : +33 (0)5.56.84.25.75
Fax : +33 (0)5.56.84.86.47

www.gretha.fr

Cahiers du GREThA (derniers numéros)

2011-03 : MARTIN Jean-Christophe, POINT Patrick, *Economic impacts of development of road transport for Aquitaine region for the period 2007-2013 subject to a climate plan*
2011-04 : BERR Eric, *Pouvoir et domination dans les politiques de développement*
2011-05 : MARTIN Jean-Christophe, POINT Patrick, *Construction of linkage indicators of greenhouse gas emissions for Aquitaine region*
2011-06 : TALBOT Damien, *Institutions, organisations et espace : les formes de la proximité*
2011-07 : DACHARY-BERNARD Jeanne, GASCHET Frédéric, LYSER Sandrine, POUYANNE Guillaume, VIROL Stéphane, *L'impact de la littoralisation sur les valeurs foncières et immobilières: une lecture différenciée des marchés agricoles et résidentiels*
2011-08 : BAZEN Stephen, MOYES Patrick, *Elitism and Stochastic Dominance*
2011-09 : CLEMENT Matthieu, *Remittances and household expenditure patterns in Tajikistan: A propensity score matching analysis*
2011-10 : RAHMOUNI Mohieddine, YILDIZOGLU Murat, *Motivations et déterminants de l'innovation technologique : Un survol des théories modernes*
2011-11 : YILDIZOGLU Murat, SENEGAS Marc-Alexandre, SALLE Isabelle, ZUMPE Martin, *Learning the optimal buffer-stock consumption rule of Carroll*
2011-12 : UGAGLIA Adeline, DEL'HOMME Bernard, FILIPPI Maryline, *Overcoming grape growers'pesticide lock-in*
2011-13 : BOURDEAU-LEPAGE Lise, GASCHET Frédéric, LACOUR Claude, PUISSANT Sylvette, *La métropolisation 15 ans après*
2011-14 : BROUILLAT Eric, OLTRA Vanessa, *Dynamic efficiency of Extended Producer Responsibility (EPR) instruments in a simulation model of industrial dynamics*
2011-15 : GABILLON Emmanuelle, *A Theory of Regret and Information*
2011-16 : BLANCHETON Bertrand, MARCHI Jean-Jacques, *Le tourisme ferroviaire en France, une mise en perspective historique de 1870 à nos jours*
2011-17 : BELIN Jean, HORBACH Jens, OLTRA Vanessa, *Determinants and Specificities of Eco-innovations –An Econometric Analysis for the French and German Industry based on the Community Innovation Survey*
2011-18 : MARTIN Jean-Christophe, BECUWE Stéphane, *Analyse de décomposition structurelle des émissions de gaz à effet de serre : application à la région Aquitaine*
2011-19 : ROUILLON Sébastien, *Solving the hotelling model in feedback form*

La coordination scientifique des Cahiers du GREThA est assurée par Sylvie FERRARI et Vincent FRIGANT. La mise en page est assurée par Anne-Laure MERLETTE.