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Collusion with a Greedy Center in Position Auctions

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Collusion avec un Centre Glouton dans les Enchères de Position

Résumé

Cet article vise à analyser la mesure dans laquelle l'enchère Généralisée au Second-Prix est sensible à la collusion dans un cadre où les transferts monétaires sont autorisés. Nous proposons un modèle d'enchère de position dans lequel une tierce partie facilite la collusion dans un cadre à information complète. Nous montrons qu'une collusion de premier-rang peut être atteinte selon n'importe quelles conditions de Nash. S'appuyant sur le critère d'Indifférence Locale nous trouvons que lorsque les gains collusifs sont uniformément redistribués entre les membres du cartel, l'issue de Vickrey-Clarkes-Groves devient l'unique équilibre du jeu. Les joueurs ne sont pas suffisamment incités à diminuer leur demande exprimée. Nous proposons ensuite des éléments selon lesquels une taxe sur les gains collusifs, compatible du point de vue des incitations, peut être fixée par ce nouvel agent. Nous mettons en lumière des conditions selon lesquelles les joueurs peuvent se coordonner de manière efficiente. Ainsi, nous contribuons à la littérature sur les questions de collusion dans des enchères simultanées d'objets multiples.

Mots-clés : Enchères; Publicité en ligne; Enchères de positions; Bidding ring; Cartel

Collusion with a Greedy Center in Position Auctions

Abstract

In this paper we aim at studying the sensitivity of the Generalized Second-Price auction to bidder collusion when monetary transfers are allowed. We propose a model of position auction that incorporates third-parties as agents facilitating collusion in complete information. We show that the first-best collusive outcome can be achieved under any Nash condition. Under the locally envy-free criterion, we find that if the collusive gain is uniformly redistributed among members, the best that can be achieved is Vickrey-Clarkes-Groves outcome. Bidders do not have sufficient incentives to reduce even more their expressed demand. We then provide elements upon which an incentive compatible fee can be set by the center. We provide conditions under which bidders can enhance efficient collusion. Doing so we also contribute to the literature on collusion in multiple-objects simultaneous auctions.

Keywords: Auctions; Online advertising; Position auctions; Bidding ring; Cartel

JEL: D44, C72, M3, L41

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1 Introduction

Collusion is one of the most long-standing issue for auction design as bidder collusion appears to be the most prominent threat to the seller's revenue. This notion refers essentially to any attempt to engage in anti-competitive behavior. The most natural reason is to suppress rivalry and get an extra rent that should have been captured by the seller. Therefore, the presence of a cartel or ring can have substantial effects on the auction revenue and on final allocation efficiency[1]. This practical concern has motivated a large amount of works in auction literature (experimental and theoretical) mainly since the seminal works by Graham and Marshall (1987), Mailath and Zemsky (1991), McAfee and McMillan (1992).

We are interested in bidders' collusion that actually occurs in one of the widest designed auction ever used, namely the *Generalized Second-Price* auction (*GSP*) daily run on internet by search engines. Each time a customer enters a research query, an automated auction is triggered putting advertisers in competition for ad spaces. Afterwards, each time a customer clicks on one's ad this advertiser is charged a price determined by the auction rules. As this market generates billions of dollars the question whether the *GSP* auction stands out to be robust or not to bidding rings is of high practical concern.

In addition to customers, sponsored search markets involve third-parties, which act on behalf of advertisers[2]. These *intermediary firms* undercut search engines ability to extract surplus from bidders and disrupt their bargaining power by offering new skills and market knowledge, which reduces the knowledge gap between advertisers and search engines. We can see them as organizing collusion among the bidders since they explicitly enhance coordination of advertisers' bids and this paper aims to analyze the extent to which the *GSP* can be conductive to collusion and under which conditions bidders can implement an efficient collusive mechanism.

Our analysis will provide elements as regards to the following points:

- The sustainability of collusion whenever the *intermediary firm* seeks to exploit spoils of collusion while compressing within competition.
- The *demand-reduction* as a persistent phenomenon.
- The maximal revenue that a seller can extract in presence of a cartel.
- *VCG-equivalent* outcome as the result of coordination.

In a model with two positions and three players we show that indeed (i) the *intermediary firm* can coordinate bidders on an *efficient* equilibrium which entails higher surplus than the *lowest*-Nash equilibrium by pinning down the outcome to a *low-revenue* equilibrium (propositions (3) and

[1] If an auction is designed so that bidders have incentives to set up a bidding ring, nothing ensures that the seller will effectively capture the second-highest valuation among players as the effective payment at the end of the game.

[2] The aim of such firms boils down roughly to minimizing advertisers' bids subject to maximizing conversions from costumers. We are not dealing with the magnitude of this optimization or by the extent to which firms could obtain more conversions upon few clicks.

(5)) and (*ii*) it can extract a fee upon the spoils (proposition (6)) without breaking incentives. In addition, it is shown that if the center was to become absolutely greedy the unique collusive outcome becomes the *lowest*-Nash equilibrium. This implies an upper boundary on bidders' payoffs (proposition (7)). The *non-cooperative* outcome can thus be sustained as a collusive one.

Usual findings in the *non-cooperative* literature show that in equilibrium bidders have incentives to shade their bids with respect to their true valuations (see proposition (1) and corollary (1)) and this shading decrease when competing for higher positions. We find an opposite relation in presence of a ring so that the more positions are different in clicks, the more is the shading. Whenever the click ratio drops to zero we end-up to the standard results of collusion in single-unit auctions (eg. Graham and Marshall (1987), Mailath and Zemsky (1991), McAfee and McMillan (1992)), in which only the highest-valuing member is active while all other members refrain from bidding. In contrast, the more positions are substitutes, the less is the shading. However, if we change the payment rule within the cartel, under mild-conditions upon the fee we find that the center can implement a truthful equilibrium. As a result bid shading vanish totally.

Given this relation, under the *non-cooperative* play the seller can extract more rent by increasing the gap in clicks. In contrast, in the present work we learn that in the presence of coordination, the seller should employ an opposite strategy. Since collusive bidding functions are non-decreasing in the substitutability of positions, decreasing this gap implies a lower level of shading. However, we observe that the auction revenue is upper-bounded by the one achieved in the lowest *non-cooperative* outcome (proposition (8)).

Finally, from the seminal works by Aggarwal, Goel and Motwani (2006), Lahaie (2006), Varian (2007), Edelman et al. (2007) it is well know that the *GSP* auction implements a *VCG-equivalent* outcome[3]. An equilibrium that has been shown to be unique if the game is analyzed as an ascending-price auction (e.g Edelman et al. (2007)). Nevertheless, it is unclear whether the *GSP* auction is effectively run as a simultaneous ascending-price mechanism and whether it will end-up to such outcome. We offer a justification of why bidders should end-up to the *lowest-symmetric* Nash equilibrium. Indeed, if side payments were to be based on the *locally envy-free* stability criterion then introducing a third-party rules out the multiplicity of equilibria and the *VCG*-equivalent outcome becomes the unique ending point (proposition (9)). This observation sustains the robustness of Varian (2007), Edelman et al. (2007) stability criterion and the robustness of the *GSP* to collusion under this criterion. Bidders do not have sufficient incentives to reduced their expressed demand and the best that can be done is the *non-cooperative* outcome.

1.1 Related Literature

The literature main focus is on the competitive framework and the justification of the *VCG*-equivalent outcome. To the best of our knowledge no work has investigated the issue of explicit coordinated behaviors. For instance, akin to the *VCG* justification, Börgers et al. (2013), based

[3]In this state, the final allocation achieved is *efficient*, maximizes social welfare and the vector of equilibrium prices constitutes the minimum competitive market-clearing prices

on the work by Milgrom (2000) (theorem 3), have shown in a broader setup the strict analogy between the stability criterion retained by Varian (2007), Edelman et al. (2007) and the walrasian *tatônnement* process which offers a natural explanation of the *VCG* outcome. Based on the idea that price anticipation is a difficult task to pursue, Cary et al. (2007) have shown that if we consider that bidders uses myopic best-response[4] then indeed the mechanism converge to the *lowest-competitive* equilibrium. Using the mechanism design approach, Edelman and Schwartz (2010) find out that in the *GSP* auction with a unique optimal reserve price, the *lowest-competitive* equilibrium is the only one that survives the *"non-contradiction criterion"* . However, it seems that the standard *non-cooperative* setup is not robust to small perturbations such as stochastic entry. For instance, Athey and Nekipelov (2012) use some noise in bidders' estimations of click-through rates and also allow them to randomly disappear during the auction. They find that their pure strategy equilibrium may not be *VCG*-equivalent. Similarly, by introducing noisy (or dummy) bidders randomly during the auction process, Hashimoto (2013) find that *VCG*-equivalent equilibria fail to exists, so as pure strategy equilibria, as the probability of entry grows.

We take the counterpart of the NC play and learn that by nesting the setup to a particular collusive device, the only possible ending point becomes the *lowest-Nash* equilibrium. This outcome generates an upper boundary on bidders' payoffs. To the best of our knowledge only two papers offer an explicit representation of coordinated bidding in two different frameworks.

One is by Ashlagi et al. (2009) explicitly implement a coordinating device to the position auction through the solution concept of *mediated equilibria*. They consider Bayes-Nash equilibria and introduce a third-party, called a *mediator*, which by using the revelation principle implements the VCG outcome as the unique possible one. They extend this result to a broader class of position auctions including the *generalized first-price* auction and the *generalized* m^{th}*-price* auction. This mediator acts in the same way of the third-party we introduce in the present work with the restriction of no side contracts. We depart from their analysis as we allow for transfers among bidders despite the similar result we obtain. Ours stems from the application of the *LEF* criterion along with some position-dependent side-payments. Moreover, we find it more realistic to assume that the intermediary firm gives a monetary payback to each advertiser with whom it has a bilateral contract.

The other one, by Vorobeychik and Reeves (2008) consider collusion in the context of repeated position auction in a different way of the above authors. Based on the work by Cary et al. (2008) and the myopic best-response they consider, Vorobeychik and Reeves (2008) shows that there exists some collusive equilibrium supported by strict *non-cooperative* behaviors. In their equilibrium, each player bids an infinitesimal quantity above the valuation of the first player not assigned to the list of positions and this equilibrium strategy profile is supported by a specific punishment threat. In case of a defection from a bidder, the remaining one automatically trigger the *VCG* equilibrium profile which implies a strict loss for the defector. In contrast to this work, our model suggests that

[4] That is they choose their optimal strategy for the next period based on the conjecture that their competitors will not modify their bids.

except from the limit case cited above, in any collusive equilibrium it is optimal for each member to bid a quantity strictly below the valuation of this low-valuing player (propositions (3) and (5)). It has to be noted that contrary to Vorobeychik and Reeves (2008) the collusion takes place in a one-shot GSP auction with monetary transfers and without any grim-trigger strategy.

The *GSP* mechanism is also related to the simultaneous ascending-price auction which is experienced to be highly conductive to collusive behaviors[5]. However, it is not fully understood whether this selling mechanism imports the weakness to coordinated behaviors inherent to simultaneous ascending-price auctions in general (Kwasnica (1998), Milgrom (2000), Ausubel et al. (2014)). We find that the *low-revenue* property of the simultaneous ascending-price auction, established by Engelbrecht-Wiggans and Kahn (2005) and Milgrom (2000)[6], also emerges at the *first-best collusive outcome* of the *GSP* auction. However, it should vanish whenever the positions are seen as perfect substitutes and/or whenever the center was to become too greedy on extracting a rent from the collusion (propositions (3), (5) and (7))[7].

The issue of *demand reduction* has been analyzed in detail by Ausubel et al. (2014) in the context of auction with multiple identical objects when buyers have non-increasing marginal utilities. They show that buyers have incentives to shade their bids with respect to their marginal utility for each subsequent units as their own bids affect the price they pay on each unit at the end of the auction. This pathology is akin to the multi-demand assumption and should vanish if buyers were to ask for one unit at the most. However, if we consider that bidders have increasing and highly different marginal willingness to pay for each unit, then Brusco and Lopomo (2002) are able to show that even in the multi-demand assumption the rational for *demand reduction* disappears as each is better off winning an entire bundle of goods instead of coming to a *split awards* agreement.

Despite the unit-demanders assumption, *demand reduction* is a persistent phenomenon in the *GSP* auction as in equilibrium it is not optimal to bid one's own true valuation for a click. There exists a tight link between the click ratio over positions and the competition nature between players. If the gap in clicks is so that the top position is the most attractive then the game reduce strategically to a standard Vickrey auction where it becomes optimal to be one's own valuation. In contrast, if the gap in clicks is so that positions becomes perfect substitutes then the shading drops to an extreme level as the game results in a standard Bertrand competition.

In the present work however, if positions are quite different then incentives converge to those observed in collusion in standard second-price auction when side contracts are allowed as the competition is essentially about the top position and in contrast cannot reaches a lower level than the one in the *lowest* Nash equilibrium. Hence, under our specific context the phenomenon of *demand reduction* is pathological and permanent even if bidder are single unit-demanders. Nevertheless, if

[5]See Cramton and Schwartz (2000, 2002), Brusco and Lopomo (2002), Milgrom (2000)

[6]Milgrom (2000) has shown under the assumption of common knowledge valuations that a simple sequential equilibrium resulting in the lowest prices allowed by the auction exists. The outcome reaches the smallest equilibrium possible but is inefficient as units are split between buyers and are not won by the highest-valuing one. See theorem 7-8.

[7]This result also asserts the idea that in complete information the standard results by Graham and Marshall (1987), McAfee and McMillan (1992) can be export to this particular auction mechanism.

in our model the center set up a redistribution rule so that individual contributions become conditional to each member deviations, then rational for bid shading disappears as this agent is able to implement and restore an outcome in which telling the truth is an equilibrium.

This paper analyzes a model of collusion applied to the context of position auctions in which (1) the third party organizes a ring by approaching all competing advertisers on the same keywords and negotiates with them (2) it offers specific individual contracts composed by monetary transfers and the amount it intends to keep from the coordination surplus (3) the auctioneer does not implement rules that could destroy incentives to collude. As regards to the all-inclusive cartel assumption, it appears for us to be a sufficient condition to analyze efficient and profitable collusion. Such environment is optimal as the intermediary firm will have to only control for its members' bids and not to anticipate bids of potential outside bidders. Then, if a profitable and efficient collusion cannot be implemented under the all-inclusive assumption one can reasonably assume that it will not be the case if we allow for incomplete cartels.

The paper is organized as follows. Section (2) describes the model, in which we introduce collusive device involving side-payments by following Graham and Marshall (1987) and the main issue the *intermediary firm* faces. It then presents the *non-cooperative* framework with an application to our simple context of two positions and three bidders. Finally, it exposes the link between competition and substitutability between positions. In section (3) we split the analysis between a collusion implemented by a center who acts as a social planner and then becomes active by levying a strict positive fee upon the collusive spoils and present our main results. Section (4) offers a justification of the *VCG*-equivalent outcome as a unique ending-point.

2 The model

In this section we introduce the collusion model akin to the position auction game, which follows from Varian (2007), Edelman et al. (2007), present the non-cooperative equilibria relevant to our setup, enlightened the demand-reduction phenomenon in this case and the coordination issue faced by the third-party.

2.1 The position auction game with collusion

We model the position auction as a game involving an all-inclusive cartel (or *ring*) of n bidders from a set $\mathcal{N} = \{1, \cdots, i, \cdots n\}$ who compete for m positions from a set $\mathcal{K} = \{1, \cdots, k, \cdots m\}$ simultaneously sold by a search engine and a third-party who plays the role of what we call the *intermediary firm* or *center*. We assume that $n \geq m$.

Each position k returns a commonly known expected *click-through rate* (*ctr* so forth) denoted by $\alpha_k \geq 0$ which are assumed to be ordered so that $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ with $\alpha_{m+1} = 0$ by convention. In this game, a player's valuation expresses his willingness to pay for a click and is denoted by x_i, with $x_i \in [0, \bar{x}]$. Let $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ be the set of individual valuations labeled in decreasing order so that $x_1 > x_2 > \ldots > x_n > 0$. We assume that valuations are independent from positions, identities of advertisers allocated[8] and of customers clicking behaviors[9]. So forth, i 's value of being allocated in position k is defined by the product $\alpha_k x_i$.

In the auction game, each player will simultaneously choose some bid b_i for a click as a function of his own valuation. We define a *pure strategy* for player i to be a function $\beta_i : [0, \bar{x}] \longmapsto \mathbb{R}_+$ assigning a uni-dimensional and non-negative bid $b_i = \beta_i(x_i)$ to each type x_i. To distinguish between a *non-cooperative* (or *Nash*) bid and a *collusive* one we denote by b_i^{gsp} and by $b_i^{\mathcal{N}}$ the former and the latter respectively. Then, let $\boldsymbol{B} = \{b_1^{gsp}, \ldots, b_n^{gsp}\}$ be the set of bids and $\mathbf{B}_{-i}$ the set of bids except b_i^{gsp} and let $\boldsymbol{B}^{\mathcal{N}}$, the set of possible actions for ring members and $\boldsymbol{B}_{-i}^{\mathcal{N}}$ defined in a similar fashion.

Basically, the game consists in allocating positions to bidders based on the order of their bids (or *expressed demand*). An allocation will thus be an assignment formally defined by a function $\iota : \mathcal{K} \mapsto \mathcal{N}$ assigning a position $k \in \mathcal{K}$ to some bidder $\iota(k)$. Then, according to the rule of the *GSP* once a player $i \in \mathcal{N}$ is assigned a given position $k \in \mathcal{K}$ he will be charged a price *per* click $p_{i,k}^{gsp}$ equal to the bid of the player ranked just below him. Thus, the net expected *non-cooperative* payoff of a player $\iota(k) \in \mathcal{N}$ if assigned a position $k \in \mathcal{K}$ will take the following quasi-linear formulation:

$$\pi_{\iota(k)}^{gsp} = \alpha_k \left(x_{\iota(k)} - p_{i,k}^{gsp} \right) = \alpha_k \left(x_{\iota(k)} - b_{\iota(k+1)}^{gsp} \right) \tag{1}$$

Now that the basics of the competitive play are settled, consider that bidders meet before the main auction starts. If they all attempt to outbid each other during the main auction they will give most of the surplus to the seller. The task is then to find an agreement to limit the surplus extraction and to expropriate a significant share of it from the seller. The objective of the *intermediary firm* is to manage this coordination so that no bidder can find it profitable to shatter the settlement by reversing to some competitive behavior. The goal is thus to elaborate individual contracts γ_i so that each potential member is better off than in the absence of coordination.

We assume that the sustainability of the cartel (or *ring*) is based on the presence of side-payments. That is, there is a monetary transfer $\omega_{\iota(k)}$, corresponding to a bidder $\iota(k)$'s contribution, from each member to the *intermediary firm*. In return the *center* makes a lump-sum transfer to each member of the collusive benefits[10].

Individual contributions are based on the anticipated final allocation and supported as a *sunk cost* by members. Each has to pay his own contribution no matter the final allocation is. Those payments are computed in a standard fashion as being the difference between the price a bidder $i \in \mathcal{N}$ would have pay to the seller in absence of a *ring* and the price he actually pays with the *ring* operating.

[8]We do not treat the question of the allocative externality generated for instance by firm's reputation among customers. A high-reputed firm might imply more clicks for unknown one if the former is placed on a position just above than a low-reputed firm could have done.

[9]This is a highly simplifying assumption. It can be observed that some type of customers, more experienced, click first more carefully and more on ads placed in median position. The quality of the ad and the reputation of firms affect their choices. Then one could allow valuations for clicks to vary non-linearly between positions.

[10]To be clear, the latter is done at the end of the main auction so that the third party holds the entire bargaining power and affects necessarily the incentives compatibility constraints.

We will denote by $\tilde{p}^{\mathcal{N}}_{i,k}(\mathbf{x})$ the payment of member $i \in \mathcal{N}$ to the seller when assigned a position $k \in \mathcal{K}$ as a function of cartel's members types when the *ring* operates. Thus, the payment $\omega_{\iota(k)}$ takes the following simple formulation:

$$\omega_{\iota(k)} = \max\left\{p^{gsp}_{i,k}(\mathbf{x}) - \tilde{p}^{\mathcal{N}}_{i,k}(\mathbf{x})\,;0\right\} = \max\left\{\alpha_k\left(b^{gsp}_{\iota(k+1)} - b^{\mathcal{N}}_{\iota(k+1)}\right);0\right\} \tag{2}$$

The total expected gain $\Pi_{\mathcal{N}}$ of the collusion (referred hereafter as *spoil*), to be *uniformly* redistributed at the end of the auction, is then the sum of each member's payment to the *intermediary firm*:

$$\Pi_{\mathcal{N}} = \sum_{k\in\mathcal{K}} \omega_{\iota(k)} = \sum_{k\in\mathcal{K}}\left(p^{gsp}_{i,k}(\mathbf{x}) - \tilde{p}^{\mathcal{N}}_{i,k}(\mathbf{x})\right) \tag{3}$$

At the end of the auction, the *center* can credibly exert his bargaining power by setting a fee $\varepsilon \in [0,1]$ upon $\Pi_{\mathcal{N}}$. He can behave in three manners. Wether (i) sporting the role of a risk-neutral incentiveless agent (which we refer to the *social planer* situation) when $\varepsilon = 0$ or (ii) levying some fee $0 < \varepsilon < 1$ upon the collusive profits (which we refer to the *skewed redistribution* situation) or (iii) keeping the entire collusive gain with $\varepsilon = 1$ and substitute the seller. The quantity ε is set *non-strategically* and each member i, given allocation ι, receives a transfer τ equals to:

$$\tau = \frac{(1-\varepsilon)}{n}\sum_{k\in\mathcal{K}}\omega_{\iota(k)} = \frac{1-\varepsilon}{n}\sum_{k\in\mathcal{K}}\alpha_k\left(b^{gsp}_{\iota(k+1)} - b^{\mathcal{N}}_{\iota(k+1)}\right) \tag{4}$$

We call the individual quantity in (4) a *uniform-ε redistribution rule* and is assumed to be retained by the *center* if a defection occurs. Note that τ is independent as i's own allocation. Given the redistribution scheme, the total payment of player i to the cartel will be function $p^{\mathcal{N}}_i : \mathbb{R}^{\mathcal{N}} \mapsto \mathbb{R}^+$ so that given allocation ι individual payments take the following formulation:

$$p^{\mathcal{N}}_{\iota(k)} = \begin{cases}\omega_{\iota(k)} - \tau & \text{if } 1 \le k \le m\\ -\tau & \text{if } k > m\end{cases} \tag{5}$$

that is bidder $\iota(k)$ if assigned to position $k \in \mathcal{K}$ pays to the cartel is individual contribution and receives his individual share from the *center* is returns whereas if $k \notin \mathcal{K}$ only gets his individual share as he does not contribute[11]. As a result, a member's expected gain equals:

$$\pi^{\mathcal{N}}_{\iota(k)} = \begin{cases}\alpha_k\left(x_{\iota(k)} - \max\left\{b^{\mathcal{N}}_{\iota(k+1)}, r\right\}\right) - p^{\mathcal{N}}_{\iota(k)} & \text{if } 1 \le k \le m\\ -\tau & \text{if } k > m\end{cases}$$

[11]The environment of complete information highly simplifies the functioning of such a redistribution rule as there is no need to implement a *pre-knockout* in order to make players reveal their private willingness to pay for a click. There is no adverse selection in this framework and no issue of cartel misrepresentation at the main auction.

The surplus of the *intermediary firm* is simply equal to the following:

$$\Gamma_{\mathcal{N}}(\mathbf{x},\iota) = \sum_{k\in\mathcal{K}} \omega_{\iota(k)}(\mathbf{x}) - \sum_{i=1}^{n} \frac{(1-\varepsilon)}{n} \sum_{k\in\mathcal{K}} \omega_{\iota(k)}(\mathbf{x}) = \varepsilon \sum_{k\in\mathcal{K}} \omega_{\iota(k)}(\mathbf{x})$$

If $\varepsilon = 0$ the *center* keeps nothing from the collusion and all the profits goes to bidders, which is situation *i*). If $\varepsilon = 1$ he then retains the entire collusive profits, which is the surplus that should have been expropriate by the seller.

The *intermediary firm* offers to each bidder a contract, in which he committed to, composed by: a system of recommended bids $\boldsymbol{\mu} = \left(b_i^{\mathcal{N}}\right)_{i\in\mathcal{N}}$ to be played during the auction, the share $\varepsilon \in [0,1]$ it intends to keep and the side-payments required from each member $\mathbf{p}^{\mathcal{N}} = \left(p_i^{\mathcal{N}}\right)_{i\in\mathcal{N}}$. For the sake of simplicity, we assume that the seller is passive by setting a reserve price equal to zero and that each bidder cannot act as an intermediary firm by reselling his position afterwards.

2.2 Coordination issue faced by the center

Before the auction starts the center makes the contract proposal $\boldsymbol{\gamma}_i = \left(b_i^{\mathcal{N}}, p_i^{\mathcal{N}}, \varepsilon\right)$. Each potential member accepts the proposal or rejects it. If one rejects then the ring does not form, otherwise the ring operates. This is referred to the *participation phase.* Members are then asked to pay their contributions ω_i and to bid at the main auction. They can act according to the recommendation, i.e bid $\mu_i = b_i^{\mathcal{N}}$, or they can defect the agreement and bid some $\tilde{b}_i$, double-crossing the *ring.* This will be the *deviation phase.* The seller then allocates bidder in decreasing order of their bids and each is charged $\tilde{p}_i^{\mathcal{N}}(\mathbf{x})$. If no defection occurred during the auction stage, the center redistributes τ and nothing otherwise.

The center's problem of maximizing the collusive surplus can be written as follow: maximize the total (*ring*) surplus to the n members:

$$\max_{b_1^{\mathcal{N}},\ldots,b_n^{\mathcal{N}}} BS = \sum_{i=1}^{n} \pi_i^{\mathcal{N}}\left(b_i^{\mathcal{N}}, \mathbf{B}^{\mathcal{N}}, x_i\right) \tag{6}$$

subject to incentive compatibility, $\forall i \in \mathcal{N}$, $\forall b_i^{\mathcal{N}} \in \mathbf{B}^{\mathcal{N}}$ and $\forall x_i \in \mathbf{X}$:

$$\pi_i^{\mathcal{N}}\left(b_i^{\mathcal{N}}, \mathbf{B}_{-i}^{\mathcal{N}}, x_i\right) - \pi_i^{dev}\left(\tilde{b}_i, \mathbf{B}_{-i}^{\mathcal{N}}, x_i\right) \geq 0 \tag{7}$$

in which $\pi_i^{dev}\left(\tilde{b}_i, \mathbf{B}_{-i}^{\mathcal{N}}, x_i\right)$ represents player i's surplus if he exploits the collusion with a bid $\tilde{b}_i \neq b_i^{\mathcal{N}}$ when the other plays according to the strategy $b^{\mathcal{N}}$. Given allocation ι profits from upward and downward deviations take the following formulation:

$$\forall l > k:\ \pi_{\iota(k)}^{dev}\left(\tilde{b}_{\iota(k)}, .\right) = \alpha_l\left(x_{\iota(k)} - \max\left\{b_{\iota(l+1)}^{\mathcal{N}}, 0\right\}\right) - \omega_{\iota(k)} \tag{8}$$

$$\forall s < k:\ \pi_{\iota(k)}^{dev}\left(\tilde{b}_{\iota(k)}, .\right) = \alpha_s\left(x_{\iota(k)} - \max\left\{b_{\iota(s)}^{\mathcal{N}}, 0\right\}\right) - \omega_{\iota(k)} \tag{9}$$

and finally subject to individual rationality:

$$\pi_i^{\mathcal{N}}\left(b_i^{\mathcal{N}}, \mathbf{B}_{-i}^{\mathcal{N}}, x_i\right) \geq \pi_i^{gsp}\left(b_i^{gsp}, \mathbf{B}_{-i}, x_i\right) \tag{10}$$

The incentive compatibility constraint (7) expresses the idea that once he receives his recommended bid from the *center* (prior playing at the auction), bidder i has the choice between obeying and adjusting his bid against what his competitors are asked to play at the auction[12]. If deviation occurs, the defector does not receive his individual compensation τ but still incurs his *supposed efficient* contribution as a *sunk cost*. This constraint requires that there does not exist a deviating bid $\tilde{b}$ that allocates bidder i any different position so that he is better off. The last constraint means that player i finds it profitable to join the *ring*.

Definition 1. We will say that a collusive mechanism $\boldsymbol{\zeta} = \left(\boldsymbol{\mu}, \mathbf{p}^{\mathcal{N}}\right)$ is an equilibrium profile if it is (i) individually rational and (ii) incentive compatible.

Usually a collusive device is said to implement an efficient outcome if the highest-valuing member of the collusion represents the cartel, wins the object and the ring is able to suppress within-competition. Such an outcome naturally maximizes social welfare as the good is allocated to the highest-valuing player. We will slightly modify this definition to make it coherent with the position auction context.

Definition 2 (*First-best outcome*)**.** We will say that a collusion in the position auction is an efficient mechanism if:

1. Final allocation is such that only the k^{th}-highest valuing members are active during the targeting auction and all other members refrain from bidding.

2. It suppresses competition from bidders not allocated to any position.

3. It maximizes social welfare defined as the quantity[13]:

$$\mathcal{W}(\mathbf{x}, \iota) = \sum_{j \in \mathcal{K}} \alpha_j x_{\iota(j)}$$

That is, if positions are allocated in decreasing order to the valuations, i.e $\forall i : \iota(i) = i$.

The multi-objects auction framework of the position game implies that the standard definition of an efficient outcome cannot be strictly implemented. It would mean that only the highest-valuing member stays active during the target auction whereas all other players refrain from bidding or bid the lowest possible increment allowed by the rule of the auction. Doing so implies that $m-1$ players among the $n-1$ will randomly be assigned to the position set which creates an outcome naturally socially sub-optimal.

[12] Assuming they behave according to the recommendation.

[13] $\mathcal{W}(\mathbf{x}, \iota)$ can be thought of as a utilitarian welfare function which is obviously maximized whenever the condition $x_{\iota(1)} > x_{\iota(2)} > \cdots > x_{\iota(m)}$ is satisfied. Therefore, an equilibrium ranking ι will be *efficient* only if players are assigned according to their indices. As we order bidders in decreasing order of their valuation this implies that the function ι has to be the identity mapping.

2.3 Competitive outcome with $n = 3$ and $m = 2$

We briefly present the *non-cooperative* equilibrium outcome relevant to our setup. Let's consider a position auction with $\mathcal{I} = \{1, 2, 3\}$ and $\mathcal{K} = \{1, 2\}$ and assume that each bidder plays respectively b_1, b_2, b_3 and that valuations are ordered so that $x_1 > x_2 > x_3$. We want to construct an equilibrium strategy profile of the *GSP* auction in this setting. Definition (3) in appendix (A.1) allow us to characterize the set of Nash equilibria with 3 bidders and 2 positions described in the following proposition:

Proposition 1. *With 2 positions and 3 bidders, the static generalized second-price auction with complete information has multiple Nash equilibria characterized by the following equilibrium strategy profile:*

$$\begin{aligned} b_1^{gsp} &\in \left[x_2 - \frac{\alpha_2}{\alpha_1}(x_2 - b_3^{gsp}) \, ; \, \bar{x} \right] \\ b_2^{gsp} &\in \left[\max\{x_3 \, , \, b_3^{gsp}\} \, ; \, x_1 - \frac{\alpha_2}{\alpha_1}(x_1 - b_3^{gsp}) \right] \\ b_3^{gsp} &\in \left[\max\left\{ 0 \, , \, x_1 - \frac{\alpha_1}{\alpha_2}(x_1 - b_2) \right\} \, ; \, \min\{x_1 \, , \, x_2\} \right] \end{aligned} \tag{11}$$

Proof. See appendix (A.2). □

Each player can envision a multiplicity of best-response to each other equilibrium strategy. Equilibrium bids are indeed bounded above and below by some combinations of the lowest bidder's bid and valuations of the player above and below him and of the maximum bid support for the highest one. This is in the scope of the works by Varian (2007), Edelman et al. (2007). Notice that the above equilibrium set does not rule out inefficient and non-assortative allocations. Indeed, one can set the respective bids of player 1, the highest-valuing bidder to his lower boundary and of player 2 to his upper boundary inducing an asymmetric allocation. Actually, the only allocation ruled out is the one in which the highest-valuing advertiser wins no position and the lowest-valuing the top position[14]. One can also note that overbidding could be, in this framework, one candidate for a Nash equilibrium. Hence, the above inequalities does not restrict attention to undominated strategy.

This set will be used throughout the work for computing the collusive bid set compatible with an equilibrium at bidders' lowest Nash boundaries. The following observation gives a simple necessary and sufficient condition in order to restrict attention only on *efficient* equilibria:

Proposition 2. *In this simple framework with $n = 3$ and $m = 2$ an equilibrium in which bidder i wins position i with $i = 1, 2, 3$ exists if and only if the following condition holds:*

$$(\alpha_1 x_1 - \alpha_2 x_1) \geq \alpha_1 x_3 - \alpha_2 x_2$$

[14] Indeed, if the highest-valuing player was to win no position then it should be the case that $b_3 \geq x_1 \geq x_3$ which cannot be an equilibrium profile. If now the lowest-valuing player was to win the top position then it should be the case that $p_3 = b_1 > x_3$ implying a strict loss for him.

Proof. See appendix (A.3). □

Evenly, we can curtail the multiplicity of equilibrium by using the refinement that $\forall k > m$ $b_k = x_k$. For all possible positions which are not in the allocative set, bidders bid at least their valuations to be on the position list. Without loss of generality, we can also weakly state that for all positions $k > m$ players are indifferent between all bids between 0 and their valuations. Only the lowest bidder will use a dominant strategy and we obtain the following:

$$\begin{aligned} b_1^{gsp} &\in \left[x_2 - \frac{\alpha_2}{\alpha_1}(x_2 - x_3)\,;\, \bar{x} \right] \\ b_2^{gsp} &\in \left[x_3\,;\, x_1 - \frac{\alpha_2}{\alpha_1}(x_1 - x_3) \right] \\ b_3^{gsp} &= x_3 \end{aligned} \tag{12}$$

From Varian (2007) and Edelman et al. (2007) it is well-known that the *VCG* outcome corresponds to the lowest equilibrium point among all *efficient* equilibria. Given proposition (2) and lemma (4) in appendix (A.1) the *VCG* profile in our simple framework takes the following form:

Corollary 1. *The strategy profile* $\mathbf{b}^{vcg} = (b_1^{vcg}, b_2^{vcg}, b_3^{vcg})$ *forms a symmetric Nash equilibrium of the 3-players and 2-positions auction game, in which* b_i^{vcg} *is defined as follow:*

$$\begin{aligned} b_1^{vcg} &> b_2^{vcg} \\ b_2^{vcg} &= x_2 - \frac{\alpha_2}{\alpha_1}\left(x_2 - b_3^{vcg}\right) = \frac{1}{\alpha_1} P_1^{vcg} \\ b_3^{vcg} &= x_3 \end{aligned}$$

Under $\mathbf{b}^{vcg}$ *the allocation is efficient and generates a revenue to the seller of* $R^{vcg} = x_2(\alpha_1 - \alpha_2) + 2\alpha_2 x_3$.

Proof. Using definition (4) and applying the *locally indifference* condition, the result is immediate. □

From a revenue perspective, *VCG* generates a revenue so that any Nash equilibria produces a revenue bounded below at least by some constant factor of it (see Lucier et al. (2012) theorem 5).

Notation 1. Two equilibria of the GSP in this framework reside at the boundaries of the Nash set we will referred to as *Lower-Nash Equilibrium* (LE) and *Upper-Nash Equilibrium* (UE). Each profile will be denoted respectively by $\mathbf{b}_L^{gsp} = \left(b_{L,i}^{gsp}\right)_{i \in \mathcal{I}}$ and $\mathbf{b}_U^{gsp} = \left(b_{U,i}^{gsp}\right)_{i \in \mathcal{I}}$. We will call the equilibrium achieved at lower boundaries of (12) the *Dom*-Nash equilibrium denoted by $\mathbf{b}_{Dom}^{gsp} = \left(b_{Dom,i}^{gsp}\right)_{i \in \mathcal{I}}$ and the one of corollary (1) by $\mathbf{b}_{sym}^{gsp} = \left(b_{sym,i}^{gsp}\right)_{i \in \mathcal{I}}$.

Remark 1. Finally, among all possible Nash allocation we can observe that if we let $\mathbf{b}^{gsp}$ be an equilibrium such that $\mathbf{b}^{gsp} \geq \mathbf{b}_L^{gsp}$ then $\mathbf{b}^{gsp}$ is Pareto dominated in terms of payoffs by $\mathbf{b}_L^{gsp}$.

Any strategy profile of the *GSP* auction gives necessarily lower payoffs to bidders than the lowest possible Nash equilibrium, which is rather a straightforward fact.

2.4 Link between competition and clicks ratio

Demand-reduction and the inefficiency observed in standard multiple objects auctions essentially relies on the multi-unit demand assumption. It vanishes if we restrict demand functions to be single-unit. Notice the special feature in the competition nature and equilibrium predictions of the *symmetric* Nash equilibrium from corollary (1). It is optimal for each player to bid a quantity strictly below its own valuation. Interestingly, we clearly observe from advertisers a bid shading even if they demand one object, position, at the most and that this shading tends to be lower when we move for higher positions thus for positions with higher *ctr*.

Actually, this phenomenon can be deepened or crushed upon the differentiation in clicks between positions. This enlighten us about the existence of a tight link between *ctr* and the competition nature in the *GSP* auction.

Remark 2. The relation between the ratio $\theta = \frac{\alpha_2}{\alpha_1}$, the equilibrium bids and payments is summarized in the following observations:

(i) As $\theta \mapsto 0$, bid shading vanishes. The game converges essentially to a standard 2^{nd}-price auction.

(ii) As $\theta \mapsto 1$, bid shading increases. The game essentially converges to a *Bertrand* competition.

Take the set of Nash equilibrium bids in both (11) and (12) and observe that bids are convex combinations weighted by the ratio θ.

To see (i) take the restricted set in (12). As $\theta = 0$ the first position becomes the only object that is worth to win (the most attractive one). In this case, equilibrium bids becomes bounded by the next-to-top valuation and the next-to-bottom one. The link is evenly more tight at the *symmetric* equilibrium in which indeed bidding one's valuation becomes the only possible bid as it is the case in standard 2^{nd}-price auctions.

Now for (ii), note that as $\theta = 1$, both positions becomes substitutes and the mid-valuing player's equilibrium bid becomes equal to x_3. There is no opportunity for the highest-valuing player to undercut him and the lowest-valuing advertisers set the market-clearing price to his private valuation. This situation enlighten the analogy with a Bertrand competition and the *demand-reduction* phenomenon well-known in standard multi-object auctions[15].

3 Main results

In this section we present the main results of the present work. We first consider the case of a neutral *intermediary firm* and learn that it is optimal for bidders to always bid to a level below the

[15] See Engelbrecht-Wiggans and Kahn (1998), Ausubel et al. (2014) for instance

valuation of the low-valued member. We then proceed with an active *intermediary* and find that there exists a threshold up to which: (*i*) he increases his profits without destroying the profitability of the cartel and (*ii*) he suppresses competition from the low-valued member. Finally, we treat the question of the auction revenues for the seller and learn that those are upper-bounded by the *lowest-Nash* equilibrium outcome.

3.1 The social planner case

We begin by considering that the *center* behaves in a social planner fashion and computes individual contributions conditionally to the theoretical efficient Nash allocation deduced from the set (11). That is we assume that contributions are independent of the allocation reach at the end of the main auction and computed *ex-ante.*

In this case, the *intermediary firm* acts as an incentiveless and credible banker, to whom bidders entrust their expected cash surplus from colluding *efficiently.* The fee is set to $\varepsilon = 0$ so that the entire surplus is reallocated among members. Here, even if playing on their behalf is rather impossible or limited the *center* of the *ring* will implement minimal but sufficient penalties to deter defection.

Lemma 1. *If the click ratio of position two over position one* $\theta = \frac{\alpha_2}{\alpha_1}$ *is such that* $\theta > \frac{b_2^{gsp}}{3x_3 + b_3^{\mathcal{N}} - b_3^{gsp}} = \eta$ *and if* $3\alpha_2 \geq \alpha_1$ *then the lower boundary of the second-highest valuing member is strictly positive. If the click ratio of position two over position one* θ *is such that* $\theta < \frac{3x_1 - 4b_2^{\mathcal{N}} + b_2^{gsp}}{3x_1 - b_3^{gsp}} = \rho$ *then the lower boundary for the lowest-valuing member equals 0.*

Proof. See appendix (A.4). □

This lemma says that when the *click-through rates* of both positions become close enough then the second-highest valuing member becomes more aggressive and makes the competition between both player 1 and 2 more stringent. It follows that the center can compress competition to a close-to-zero level. The mechanism results in the highest-valuing *ring* members bidding at the auction stage and suppress the bid of the low-valued member.

Proposition 3. *For the GSP auction and under the assumption that* $\theta > \eta$*, there exists an efficient equilibrium collusive profile that constrains within-cartel competition to a level below the lowest valuation.*

Proof. See appendix. □

As in the *non-cooperative* framework, *ring* members can envision multiple best-responses against each other equilibrium collusive strategy and may allow for asymmetric equilibria (see set (1) in appendix). For instance, it will still be an equilibrium if the highest-valuing member bids at his lower boundary and the second-highest valuing member bids at his upper one and the last player bids zero.

Proposition (3) asserts that a *ring* can achieve its *first-best outcome* of definition (2) if the degree of substitutability between both positions is sufficiently high. In equilibrium, assigned *ring* members bids at the main auction a quantity strictly below the valuation of the non-assigned member who refrains from bidding (see figure (5) in appendix) which contrasts the result by Vorobeychik and Reeves (2008)[16]. Its proof is by construction and assumes that contributions are determined by the *efficient* assignment.

Although the *center* cannot prevent *ring* members from cheating[17], the mechanism involves appropriate incentives for members to follow contract γ_i. The game is construct so that it is incentives compatible for each winning bidders to make their transfer payments *ex-ante.* First, members make their required payments ω_i to the *intermediary firm*. After the auction stage, the *center* makes payment τ to each member. If one has defected then no payment is made and the defector supports ω_i as a *sunk cost* making him strictly better off by obeying.

Remark 3. Note that collusive bids $b_i^{\mathcal{N}}$ are decreasing functions in the *non-cooperative* play and spoils increase with it. As a result, the auction revenue with is higher as players are assumed to bid low if they were to play in the *non-cooperative* game[18]. To compensate the shift in individual surplus, bidders have to increase their collusive bids which allows the seller to extract a strict positive growing surplus. A comparison between payoffs and revenues for each framework is depicted in table (1) of appendix (B.2).

Finally, we can answer to a standard conjecture that is commonly observed in standard multi-object auction literature. An efficient collusion and a *low-revenue* equilibrium become unsustainable as the number of *ring* members grows while the number of objects remains fixed.

Proposition 4. *Individual collusive profits converge to* non-cooperative *profits as the size of the ring grows while maintaining fixed the number of positions:*

$$\pi^{\mathcal{N}}_{\iota(k)} = \alpha_k \left(x_{\iota(k)} - b^{\mathcal{N}}_{\iota(k+1)} \right) - \left(\alpha_k b_{\iota(k+1)} - \alpha_k b^{\mathcal{N}}_{\iota(k+1)} \right) + \frac{1}{n} \Pi_{\mathcal{N}}$$
$$\lim_{n \to \infty} \pi^{\mathcal{N}}_{\iota(k)} = \alpha_k \left(x_{\iota(k)} - b_{\iota(k+1)} \right) = \pi^{gsp}_{\iota(k)}$$

Increasing the number of members while maintaining the number of positions destroys all collusive incentives. As the number of participants grows, the spoils is divided among a growing number of non-contributors. Thus colluding becomes less attractive and each member has incentives to become more aggressive. As a result, bidding behaviors converge to those of the *non-cooperative* level.

[16]In their model, a collusive equilibrium profile emerges as a process of *non-cooperative* behaviors when considering the repeated *GSP* auction. It is an equilibrium to bid, for each player within the position set, an *epsilon* above the valuation of the first not being assigned a position and all the others play truthfully. If one deviates from the collusive strategy then all trigger the minimum revenue *symmetric* Nash equilibrium strategies. Here, coordination is done without the help of some *intermediary*.

[17]In contrast to the standard literature, the main issue is the inability of the center to provide sufficient control over bidders' bidding behaviors and deter *shill bidding* and *fake alias,* which are issues inherent to online auctions.

[18]Denotes by $R^{\mathcal{N}}$ and R^{gsp} the auction revenue with and without coordinations respectively. The seller's revenue under collusion drops to $R^{\mathcal{N}} = \frac{\alpha_1}{3\alpha_2 - \alpha_1} 3\alpha_2 x_3 - \frac{\alpha_1}{3\alpha_2 - \alpha_1} R^{gsp}$ and decrease with R^{gsp}.

Remark 4. There is no collusive equilibrium that *Pareto-dominates* any other when $\varepsilon = 0$.

The point of this section was to show that in such a market structure, with the help of an incentiveless third-party, advertisers are able to coordinate themselves to a better equilibrium price vector while retaining the same Nash allocation. This could explain why such a market structure (with this tripartite configuration) still prevails in *ad* auctions. The question remains whether given this framework this *firm* can expropriate some surplus from the collusion or not and still maintain the efficiency of the collusion.

3.2 Skewed redistribution of the spoils

We shall now consider the case in which the *center* of the *ring* stops being a neutral and incentiveless agent. He now imposes a fee $0 \leq \varepsilon \leq 1$ when making his transfer payment τ at the end and collects a profit from organizing the coordination. Again, in order to achieve the best possible level of surplus, the *center* recommends optimal bids so that final allocation would have still been the same in the *non-cooperative* play. We can construct the same condition over the ratio of *ctr* when now the intermediary becomes an active player.

Lemma 2.

- *If* $\theta > \frac{b_2^{gsp}}{3x_3+(1-\varepsilon)\left(b_3^{\mathcal{N}}-b_3^{gsp}\right)} = \eta'$ *and if* $3\alpha_2 \geq (1-\varepsilon)\,\alpha_1$ *then the lower boundary of the second-highest valuing member is strictly positive.*
- *If* $\theta < \frac{3x_1+(1-\varepsilon)b_2^{gsp}-(4-\varepsilon)b_2^{\mathcal{N}}}{3x_1-(1-\varepsilon)b_3^{gsp}} = \rho'$ *then the lower boundary for the lowest-valuing member equals 0.*

This lemma straightforwardly extends lemma (1) to the situation in which the *intermediary firm* expropriates the surplus from members. When the gap in clicks between the top position and the lowest position becomes sufficiently low (i.e α_1 and α_2 becomes close enough) the second-valuing member becomes more aggressive, which is counterintuitive.

Indeed, one could expect the opposite effect. If the substitutability between both positions becomes perfect, bidders should bid less. The need in outbidding competitors to win the highest position decreases. This is true in the *non-cooperative* play for which in order to stay optimal, Nash equilibrium bids are decreasing in the click ratio θ. With coordination, if the substitutability between positions increases, incentive compatibility constraints become stringent. The low-valued player's profit from cheating and aiming the second position increases with θ. As a result, the *center* is constrained to released the second-highest valuing member's which in turn makes the inner-competition between this player and the high-valuing one difficult to contain. In response, the *center* is also obliged to increase the cooperative bid from the latter in order to maintain the ranking. This phenomena is also amplified by the behavior of the *intermediary firm*.

As these parameters are strictly exogenous, lemmata (1) and (2) give the actual window of opportunity for the *intermediary firm* to implement a *first-best* outcome with the lowest equilibrium prices.

Proposition 5. *There exists a threshold δ^* couples with η' and ρ' such that $\forall \varepsilon \leq \delta^*$ and $\forall \theta \geq \eta'$ a first-best outcome is achieved when it recommends the following equilibrium bids profile:*

$$
\begin{aligned}
b_1^{\mathcal{N}} &= x_3 - \frac{1-\varepsilon}{3\alpha_1}\left(w_1 + w_2\right) \\
b_2^{\mathcal{N}} &= \frac{3\alpha_2 x_3 - (1-\varepsilon)\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}\right)}{3\alpha_2 - (1-\varepsilon)\,\alpha_1} \\
b_3^{\mathcal{N}} &= \max\{0, r\}
\end{aligned}
$$

in which:

$$
\delta^* = \frac{(1+\theta)\,R^{gsp} - (1-\theta)\,(\alpha_1 - 3\alpha_2)\,x_1 - 4\alpha_2 x_3}{(1+\theta)\,R^{gsp} - (1-\theta)\,\alpha_1 x_1 - \alpha_2 x_3}
$$

is a non-decreasing function of the non-cooperative bidding functions with $R^{gsp} = (\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp})$ and $\theta = \frac{\alpha_2}{\alpha_1}$.

Proof. The proof follows from the one of proposition (3).

□

This proposition asserts that for high enough degree of substitutability in clicks, there exists a whole bundle of fees so that the mechanism results in an outcome at which the *low-price* property prevails.

The outcome of proposition (5) is *efficient* according to definition (2). The highest-valuing player is allocated the top position and the second-valuing one the second position while the lowest-valuing player refrains from bidding. Again, as in proposition (3) equilibrium bids are maintained to a level which is strictly lower than the valuation of player 3. Competition between both assigned players is constricted the lowest compatible level given both quantity ε^* and *IC* constraints. Bidders coordinates on a set of bids which constitutes the lowest possible equilibrium in price.

This proposition also gives the main window of opportunity for the center. Indeed, it says that the threshold δ^* is a non-decreasing function of the *non-cooperative* bidding functions which implies a more stringent condition over the computation of individual contributions.

Corollary 2. *With a functioning ring under a uniform-ε redistribution rule, the position game with three players and two positions is characterized by a vector of (i) monotonically non-decreasing equilibrium bidding functions in ε and (ii) decreasing in the* non-cooperative *bid functions.*

Proof. See appendix (A.6) □

It is more difficult for the center to implement a *first-best outcome* whenever the Nash bids are assumed to be played according to $\mathbf{b}_L^{gsp}$. It entails a low level of profits, a small increase in the fee can make the low-valued member set a strict positive bid which necessarily increases within-competition.

Corollary 3. *$\forall \varepsilon > \delta^*$ there does not exist a collusive equilibrium that suppresses bidders rivalry and the equilibrium collusive bids profile jumps to:*

$$b_1^{\mathcal{N}} \in \left[b_2^{\mathcal{N}}, \bar{x}\right]$$
$$b_2^{\mathcal{N}} = \frac{(2+\varepsilon)\,\alpha_2 x_3 - (1-\varepsilon)\,(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}) - (1-\varepsilon)\,(\alpha_1 - \alpha_2)\,x_1}{(2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\,2\alpha_1}$$
$$b_3^{\mathcal{N}} = \frac{x_1\,(\alpha_1-\alpha_2)\,((1-\varepsilon)\,\alpha_1 - 3\alpha_2) - (1-\varepsilon)\,(\alpha_1+\alpha_2)\,(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}) + (4-\varepsilon)\,\alpha_1\alpha_2 x_3}{\alpha_2\,((2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\,2\alpha_1)}$$

Corollary (3) says that once the fee reaches the threshold δ^*, *IC* constraints becomes binding and members' incentives reverse. The *first-best outcome* property can no longer be maintained if the center is to become too greedy as regards to the surplus appropriation. It makes the collusion crushed from the inside. Thus, using the words by McAfee and McMillan (1992) because of the existence of a bargaining power from the intermediary firm the ring in this framework "*contains the seeds of its own destruction*"[19].

This phenomenon is in a sense intuitive. (i) Once the center decides to set its fee ε to a higher level, bidding functions increase consequently, which mechanically entails a decrease in the available surplus. In return, (ii) the *center* will need to break incentives to defect for the low-valued member which implies an increase in the overall collusive bid functions.

These two correlated effects corroborate the intuitive idea that the bargaining power of the center has a substantial pervasive effect upon the durability and the sustainability of the collusion.

Remark 5. The click ratio constraints of lemma (2) are meaningless when the center sets its bargaining power over the tipping point δ^*. Indeed, the threshold couple (η', ρ') makes no sense since (i) relation (24) in appendix (A.6) holds $\forall \varepsilon \in [0, \delta^*]$. Thus, the second-valuing member's bid is strictly positive once ε reaches the value δ^* and is thus also strictly positive for $\varepsilon = \delta^* + \sigma$. Then, (ii) $\forall \varepsilon \leq \delta$ if $\theta > \eta'$ his bid is also strictly positive. Finally, once $\varepsilon > \delta^*$ we have that

$$\frac{\partial\left(\frac{(1-\varepsilon)\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}\right) + (1-\varepsilon)(\alpha_1-\alpha_2)x_1 - (2-\varepsilon)\alpha_2 x_3}{(1-\varepsilon)2\alpha_1 - (2-\varepsilon)\alpha_2}\right)}{\partial \varepsilon} \geq 0$$

which implies that $b_2^{\mathcal{N}} > 0$ independently of the value of θ and that $b_3^{\mathcal{N}}$ is also necessarily positive. Therefore $\forall \varepsilon > \delta^*$ the thresholds η' and ρ' become unbinding.

We give a representation of the equilibrium strategies in figure (1). Both bidding functions increase in the fee and are non-differentiable at threshold δ^*. We can see that from δ^* within-competition strictly increases as the lowest-valuing member's bid becomes strictly positive. The monotonicity of the equilibrium bidding functions is a straight implication of effects (i) and (ii) described above. Coordination leads to aggressive behaviors from members in order to maintain

[19] In McAfee and McMillan (1992) this issue is due to the issue of profit attractiveness from the collusion implying an entry problem.

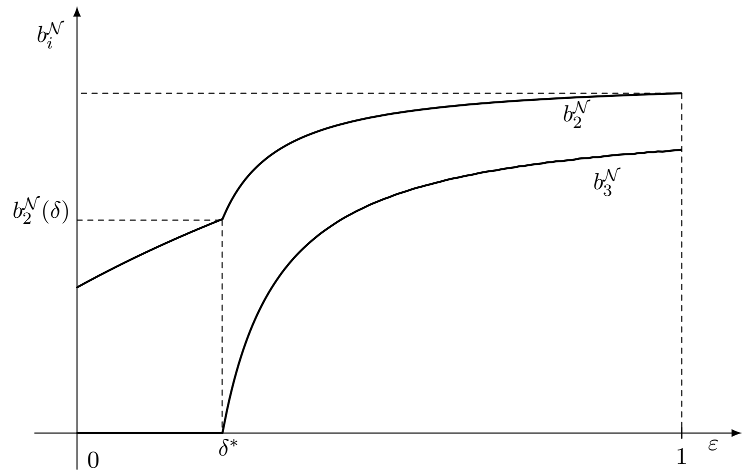


Figure 1: Collusive equilibrium bidding functions.

incentive compatibility and to compensate the downward shifting of the collusive gain in terms of individual contributions.

Remark 6. We also have $\frac{\partial \Gamma_{\mathcal{N}}}{\partial b_i^{gsp}} > 0$ and $\frac{\partial \Gamma_{\mathcal{N}}}{\partial b_i^{\mathcal{N}}} < 0$ whereas $\frac{\partial \Pi_{\mathcal{N}}}{\partial b_i^{gsp}} < 0$ and $\frac{\partial \Pi_{\mathcal{N}}}{\partial b_i^{\mathcal{N}}} < 0$ in contrast with the social planner situation. The more bidders are assumed to play high in the competitive equilibrium, the higher profits are for the center and the less joint profits. Thus both objectives are aligned in opposite ways when we integrate an active center. Doing so introduces a trade-off that the intermediary firm must cope with. In contrast to the case in which it acts as a social planners for which the cartel surplus (joint profits) is increasing in the *non-cooperative* play, giving it some bargaining power reverses the relation which is somehow a surprising implication.

Independently from the center, the seller could use this tool to deter collusion. If positions are set so that it gives the same amount of clicks to players then as collusive prices increase with this quantity of clicks, at some point it should be the case that bidders reverse to the *non-cooperative* level. This could be a reasonable guess, however it will never reach a price level higher than those in the *lowest* Nash equilibrium. Take the equilibrium bids described in proposition (5) and rearrange the expression to transform the bid of the second-highest valuing player into a function of θ. This gives the following relation:

$$b_2^{\mathcal{N}}(\theta) = \frac{3\theta x_3 - (1-\varepsilon)\left(b_2^{gsp} + \theta b_3^{gsp}\right)}{3\theta - (1-\varepsilon)}$$

If now we set the value of $\theta = 1$ then we can observe that the bid takes a quantity lower than the valuation of the lowest-valuing player. This is clearly lower than the *lowest*-Nash equilibrium $\mathbf{b}_L^{gsp}$.

Thus, even if the seller was to set arbitrarily an identical amount of clicks associated with each position, at this limit its revenue will never reach its minimal *non-cooperative* equilibrium level. This is the object of the next lemma.

Lemma 3. *A seller cannot use the ratio θ in order to induce levels of revenue higher than those of the Lower Nash equilibrium.*

In the position auction context, bidders are thus able to divide up the new available surplus in a way that does not affect incentives. Besides, we learn that such a profitable coordination can be done even if the *center* increases his monopoly power over *ring* members. It still keeps the payoff dominance of the collusion over the *non-cooperative* game. The idea of the maintained dominance relation between both mechanisms is the object of proposition (7) in the next the subsection.

3.3 Surplus expropriation

Proposition (5) allows us to characterize the maximal incentive compatible share that can be extracted from the collusion. Indeed, there exists a point up to which within-competition is contained and from which any increase in the fee has a destroying effect on the *intermediary firm*'s profits. We can thus state the following proposition:

Proposition 6. *The center's revenue is non-monotonic in ε. It is strictly increasing $\forall \varepsilon \in [0, \delta^*]$ and convex $\forall \varepsilon \in (\delta^*, 1)$. An optimal fee IC compatible is thus the point $\varepsilon = \delta^*$.*

Proof. See appendix (A.7) □

We give a representation of the intermediary profits as a function of the fee in figure (2). While bidding functions are monotonically increasing we observe that the intermediary firm is able to increase its profits. This emerges despite the decreasing gap between the *non-cooperative* function and the collusive function of the active members. The increase in the fees sufficiently compensates for the one in the collusive price which implies a lower level of available surplus for members of the collusion.

Biasing the redistribution of the surplus becomes a *plausible and valuable* strategy as it can be done without deterring the collusive incentives. Indeed, the shift in its profits arises whenever the fee becomes unsustainable for the collusion. The threshold δ^* is the critical point at which the loss in bidders' surplus needs to be compensated by increasing their respective bids. A rational behavior that mechanically implies an increase of the within-competition level.

Therefore, one of the points we wanted to highlight emerges naturally here. *The intermediary firm can always enhance a collusion with an optimal fee and this can be done without deterring any cooperative behavior. However, there exists a tipping point from which the competition between member reverses so that the collusion cannot be constrained to an efficient level according to definition* (2).

In fact, the latter is somehow a "common sense" observation as anyone could have expected this aggressive result. The level of expropriation from the *center* makes the incentives conditions more stringent for the bidders to be self-interested in the cartel formation.

Another insight offered by propositions (2) and (6) relies on the question of the sustainability of the *non-cooperative* mechanism as a limit case of any collusive mechanism. We know by proposition

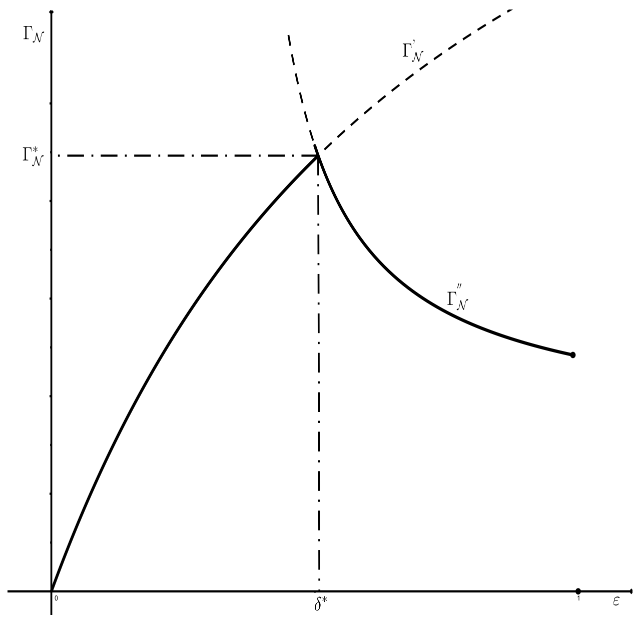


Figure 2: The intermediary firm profits as a function of ε

(1) and the restricted set in (11) that the conditions in definition (3) implements a multiplicity of Nash equilibria. Without any form of coordination, nothing guarantees a strict convergence over an equilibrium point which is Pareto-optimal from bidders' viewpoint. A natural question would be the one of coordinating bidders upon a specific one, abstracting away the issue of the lowest efficient collusive equilibria.

In our framework, the intermediary firm has the power to push upward the equilibrium prices so that bidders reverse to the *non-cooperative* behavior. Consider for a while the intermediary firm endorses the role some exogenous equilibrium "perturbation" (as some mediator in Ashlagi et al. (2009), Monderer and Tennenholtz (2009)), up to which point can this be done? The next proposition answers the first interrogation and shows that by gradually increasing the level of the fee ε (the center's bargaining power) the mechanism achieves the best Nash equilibrium from bidders' viewpoint.

Proposition 7. *The* non-cooperative *equilibrium is a sustainable collusive outcome with a uniform-ε redistribution rule. As $\varepsilon \mapsto 1$ the intermediary firm coordinates bidders over the lowest-Nash equilibrium outcome $\mathbf{b}_L^{gsp}$.*

Proof. See appendix (A.8). □

If the center expropriates all the collusive gain, i.e $\varepsilon = 1$, then outcome $\mathbf{b}_L^{gsp}$ is the unique ending point. Thus the simplest *IC* collusive mechanism which is *always feasible* stands out to be the *non-cooperative* mechanism in which players set their optimal bids consistently with the Nash equilibrium criterion.

By incrementally increasing its expropriation capability the intermediary firm is able to evict all other equilibria. See figures (1) in appendix (B.1) in which we depict the average difference between collusive bids and payoffs and those in $\mathbf{b}_L^{gsp}$. Note that at this ending point bidders' surplus are at most equal to the one achieved in $\mathbf{b}_L^{gsp}$ as $\lim_{\varepsilon \mapsto \infty} \pi_k^{\mathcal{N}} = \alpha_k \left(x_k - b_{k+1}^{gsp}\right)$. Moreover, from table (2) in appendix (B.2) in contrast with the situation in which the center acts in a social planner fashion the following observation can be made:

Corollary 4. *If $\varepsilon^* = \delta$ the center implements, from bidders' viewpoint, an outcome which weakly Pareto-dominates any other collusive outcome.*

3.4 The seller's revenue

In a nutshell, introducing the ability to extract a positive fee creates some tensions between expropriation of surplus, suppression of bidders rivalry and a *low-price* outcome. It does not induce some kind of antinomic relationship between the seller and the intermediary firm. Actually, a greedy behavior from the intermediary firm is advantageous as revenues for the seller strictly increase with the fee ε. This is the object of the next proposition.

Proposition 8. *The seller's revenue from the auction is monotonically increasing in ε. It is concave $\forall \varepsilon \in \{[0, \delta^*) \cup (\delta^*, 1]\}$ and non-differentiable at $\varepsilon = \delta^*$. The maximal quantity of surplus it can extract corresponds to the one of $\mathbf{b}_L^{gsp}$.*

In order to maintain a sufficient degree of spoils, members of the ring need to push forward the equilibrium prices which is mechanically beneficial to the seller despite the fact that this surplus has now to be divided between the players, the seller and the center.

An illustrative representation of the seller's revenue $R^{\mathcal{N}}$ (the black lines) against the overall bidders' surplus BS (the grey lines) is given in figure (3). The seller can extract most a quantity equal to the surplus generated by the $\mathbf{b}_L^{gsp}$ outcome. Indeed, as $\varepsilon \mapsto 1$ the equilibrium collusive bids converge to those sustaining the *lowest*-Nash equilibrium. Still, bidders' surplus can converge to any quantity corresponding to any outcomes associated to a particular Nash play. In the figure we denote by $\Pi^{gsp} = \sum_{k \in P} \pi_k^{gsp}$ the overall surplus achieved by the *non-cooperative* play, assuming that players are assigned efficiently, under the outcome $\mathbf{b}^{gsp}$ and by R_L^{gsp} the maximal revenue to the seller.

If the *intermediary firm* was to behave as a social planner by setting $\varepsilon = 0$ the overall surplus will be entirely divided between members and the seller, nothing is expropriated by it. He gets the minimal revenue that the presence of the efficient ring generates and members get the maximal surplus they can keep from him. Whereas, if the center was to set the maximal fee (if it intends to act as sort of *non-cooperative* equilibrium decentralizing), i.e $\varepsilon = 1$ the equilibrium bids converge to those who sustain the outcome $\mathbf{b}_L^{gsp}$, so does the seller's revenue. In all cases the seller benefits from the active behavior of the *intermediary firm*. See figure (7) for a numerical illustration in which the seller's revenue is depicted for each Nash extremum.

This positive relation between revenues and the center's greedy behavior open-up to an interesting issue that one may ask himself. Given the impossibility for a seller to artificially and indirectly drive up the collusive bids by lemma (3) and given that he strictly benefits from an increased in

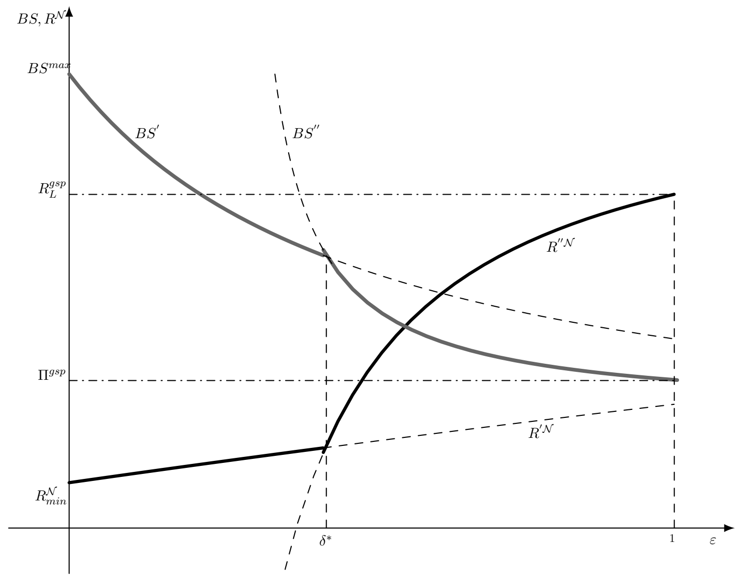


Figure 3: The seller's revenue $R^{\mathcal{N}}$ and bidders' surplus BS as a function of ε

the fee, we could naturally guess that it could be in its interest to manage some tacit agreements with the intermediary firm. In other words, this positive relationship brings up the possible and intuitive issue of a corruption device between the seller and the center. Such analysis of a corrupted seller is widely beyond the scope of the present work (cite Menezes and Monteiro (2006), Lengwiler and Wolfstetter (2010)).

3.5 Demand-reduction as a persistent phenomenon

Finally, if we compare the link between the click ratio θ as a measure of the substitutability degree of positions and the *demand-reduction* phenomenon observed in the *GSP* auction and more generally in multi-objects auctions. Usually, bid shading occurs in situations in which players' bids affect the price they have to pay at the end of the auction and should not arise whenever the same player only ask for one object at most. Here, as suggests by the equilibrium bids in (1) even if bidders are by definition unit-demanders the bid-shading is persistent in equilibrium and can be pin down to its lowest possible level if positions becomes perfect substitutes. The lowest-valuing bidder's private valuation determines the shape of the market-clearing price. However, there is still an opportunity for the seller by making the top position the only one worth having to destroy incentives to shade, which is the object of remark (2). The point when introducing coordinations and an active third agent enhancing this coordination is that the relationship between θ and the competition nature reverses and so should be the seller's strategy.

Doing so we find out an opposite effect in the substitutability of both positions and that this phenomenon is *always* recurrent at any collusive equilibrium. More specifically, if the seller is only endowed with its discrete power to affect *ctr* (as it is assumed in the present work) then bid shading cannot be reduced more than the one achieved at the *lowest*-Nash equilibrium, which implies that the revenue achieved by any collusive outcome cannot be higher than the one achieved at this *non-cooperative* outcome. On the other extreme, when positions becomes perfectly distinguishable, the 2^{nd}-price auction case, then bid shading is pushed to its extreme level and the present model nets the results by standard collusion analysis in single-unit second-price auctions with side payments as a special case (eg. Graham and Marshall (1987), McAfee and McMillan (1992)).

4 *VCG* as the unique collusive *envy-free* outcome

In this section we show that if we scale the analysis over the spirit of standard competitive equilibrium analysis by using the LEF criterion to the present framework then there does not exists a profitable way to collude except to achieve the *VCG* outcome. That is to say, if déviations are managed by members according to the standard argument of *local indifferency* then the equivalence with the *Vickrey-Clarkes-Groves* solution is restored and furthermore unique.

Let us recall that the LEF criterion implies a shift in the *IC* conditions of a Nash equilibrium so that upward and downward deviations become symmetric. Then in the spirit of Edelman et al. (2007), in order to implement a stable equilibrium bidders need to be *locally indifferent.* That is, a bidder assigned a position k when contemplating to move for position $k-1$ has to expect to pay the same price as the player in this position and to be *indifferent* between both assignments. The idea now is what if we bring such *stability condition* to the collusive game by selecting among the multiplicity of collusive equilibria the one who respects it? In other words, let us consider the LEF criteria as a *stability condition* instead of the expression of a *Walrasian tatônnement* process (as in Börgers et al. (2013)). The latter being necessarily related to a *non-cooperative* analysis.

Still assume that in case of a deviation from the collusive agreement, the defector pays his individual contribution computed *before the targeting auction starts.* That is individual contributions are computed unconditionally on deviations and based on an assortative assignement. Also, still assume that the *center* makes his transfer payment after the main auction. The *incentive compatibility* conditions are now given by the following relations: $\forall i \in \mathcal{N}$, $\forall b^{\mathcal{N}}_{\iota(i)} \in \mathbf{B}^{\mathcal{N}}$ and $\forall x_i \in \mathbf{X}$:

$$\pi^{\mathcal{N}}_{\iota(k)}\left(b^{\mathcal{N}}_{\iota(k)}, \mathbf{B}^{\mathcal{N}}_{-i}, x_{\iota(k)}\right) - \pi^{dev}_{\iota(k)}\left(\tilde{b}_{\iota(k)}, \mathbf{B}^{\mathcal{N}}_{-i}, x_i\right) \geq 0 \tag{13}$$

where $\forall k \in \mathcal{K}$:

$$\pi^{dev}_{\iota(k)}\left(\tilde{b}_{\iota(k)}, .\right) = \alpha_{k-1}\left(x_{\iota(k)} - \max\left\{b^{\mathcal{N}}_{\iota(k)}, r\right\}\right) - \left(\alpha_k b^{gsp}_{\iota(k+1)} - \alpha_k b^{\mathcal{N}}_{\iota(k+1)}\right) \tag{14}$$

This condition says that when player $\iota(k)$ assigned to position k intends to defect the collusive agreement and contemplates position $k-1$ then he expects to pay the same price as player $\iota(k-1)$ assigned this position. Hence, we give the mechanism a semblance of symmetry in the collusive equilibrium conditions and the latter results in each members bidding the *non-cooperative symmetric* equilibrium prices.

Proposition 9. *Under the locally-envy free condition and a* uniform-ε *redistribution a cartel with three players can do no better than to achieve the* non-cooperative *outcome. That is* $\boldsymbol{\mu}^*_{\mathcal{N}} = \left(b_1^{\mathcal{N}} > b_2^{\mathcal{N}}\,;\, \frac{P_1^{vcg}}{\alpha_1}\,;\, x_3\right) = \mathbf{b}^*$.

Proof. In order for player 2 to be indifferent between winning the second position at price $b_3^{\mathcal{N}}$ and winning the top position at price $b_2^{\mathcal{N}}$ the following relation should be satisfied:

$$\alpha_2\left(x_2 - b_3^{\mathcal{N}}\right) - \omega_2 + \frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} = \alpha_1\left(x_2 - b_2^{\mathcal{N}}\right) - \omega_2$$

For player three to be indifferent between being assigned to the second position and not being assigned to under the collusive agreement it should be:

$$\frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} = \alpha_2\left(x_3 - b_3^{\mathcal{N}}\right) \tag{15}$$

Rearranging both relations we obtain the following pair of equations:

$$\alpha_1 b_2^{\mathcal{N}} = \frac{3}{2+\varepsilon}\left(x_2\left(\alpha_1-\alpha_2\right)\right) + \frac{4-\varepsilon}{2+\varepsilon}\alpha_2 b_3^{\mathcal{N}} - \frac{1-\varepsilon}{2+\varepsilon}\left(\sum_{i=1}^{2} P_i^{vcg}\right) \tag{16}$$

$$\alpha_2 b_3^{\mathcal{N}} = \frac{3}{2+\varepsilon}\alpha_2 x_3 + \frac{1-\varepsilon}{2+\varepsilon}\left(\alpha_1 b_2^{\mathcal{N}} - \sum_{i=1}^{2} P_i^{vcg}\right) \tag{17}$$

Recall that $P_i^{vcg} = \sum_{k=i+1}^{m+1} x_k(\alpha_{k-1} - \alpha_k)$. Rearranging terms, plugging (17) into (16) and using corollary (1) one obtain the following relation:

$$\begin{aligned} \alpha_1 b_2^{\mathcal{N}} &= x_2\left(\alpha_1 - \alpha_2\right) + \alpha_2 x_3 = P_1^{vcg} \\ \alpha_2 b_3^{\mathcal{N}} &= \alpha_2 x_3 = P_2^{vcg} \end{aligned} \tag{18}$$

The solution corresponds for any $\varepsilon \in [0,1]$ to the same equilibrium bids and payments of the *VCG*-equivalent equilibrium bids profile of corollary (1) and thus results in the same outcome of the *symmetric non-cooperative* one. □

It is interesting to note that bidding functions are now completely independent of the fee that the intermediary firm might set. This proposition also highlights the robustness of Varian (2007), Edelman et al. (2007) stability criteria to any collusive agreements implying a lump-sum transfers of the spoils. Such observation is obviously done with respect to our setup.

5 Concluding remarks

We have analyzed a situation in which a third-party enters the *GSP* auction game and emulates an explicit form of collusion among bidders who compete for the same set of positions. Explicit collusion here means the ability to implement side-contracts in order to achieved a better outcome than the lowest *non-cooperative* one. Collusion in the present model works as follows: each bidder who wants to participate in the ring is asked an individual contribution by giving some monetary amount to the organizer. In exchange, they get at the end of the auction an equal share of the collusive surplus. The *center* makes a lump-sum transfer to each member.

We find that an efficient collusion is sustainable in equilibrium. Within competition is compressed to a level strictly below the valuation of the lowest-valuing member (the one who is not assigned to any position). Such outcome is implemented even if the third-party levies a positive fee at a threshold which entails the highest level of payoff to him. Furthermore, we find that in equilibrium bidding functions are monotonically non-decreasing in the fee, which implies that positions are allocated efficiently at the end. Among the multiplicity of collusive equilibria, we have also found that, given our framework the *lowest*-Nash equilibrium constitutes a limit case to any coordination. This equilibrium forms an upper boundary on bidders' surplus. This fact also shed lights on the maximal revenue that can be extracted by the seller. In absence of the reserve price tool the most that it can achieved is the revenue generated at this upper boundary.

As regards to the special feature of *demand reduction*, in contrast to the *non-cooperative* game we have observed a reverse relation between the competition nature and the degree of substitutability between positions. We have provided elements showing that (i) the degree of shading cannot be lower than in the lowest *non-cooperative* equilibrium and that (ii) their exists a set of fees for which the rational to reduced one's own expressed demand can be eliminated. In the mean time, based on observed market structure we propose another kind of justification to the choice of the *VCG*-equivalent outcome. We find that it is implement as a unique equilibrium as the result of explicit coordinated behaviors between bidders. To our concern, this gives an interesting consistence to the justification of the *VCG*-equivalent outcome. It is no more based solely on reasonable guesses and strengthened the *symmetric stability criterion.*

Our analysis rests on the assumption that valuations are common knowledge and that the *intermediary firm* does not suffer an *adverse selection* issue when implementing the collusive device. As one could argue, restricting attention to the complete information case might be weakly instructive as the more informative is an auction the more sustainable is the collusion. Yet, because of tractability issues the *GSP* auction does not inherit a high ground of studies in incomplete information. Therefore, we find that an adequate understanding of how collusion with side-payments works in complete information is a necessary prior to further extensions upon studying collusion in position auctions with incomplete information.

Another limitation to the present analysis stems from the equal contribution assumption. In our model, bidders gets at the end an equal share of the spoils generated by the collusion which at some extent appears to be a restrictive assumption. It would be interesting to allow for payments from the *center* to the member to be conditional on the importance that each player has within the coalition. For instance a redistribution as a function of the market power of each firm on the product market. Finally, if one assumed as we do that players are not heterogeneous, the unidimensional assumption is not compelling and it shall be reasonable to consider that they do not have further information, which could modifies their market perception and their valuation for a click. Usually this is not the case and firms have strictly more information than the seller about customers purchases behavior on their own market. For instance, individual valuations should not be consider to decrease monotonically across positions. Hence, this strong information asymmetry should be taken into account in the model as the latter is highly valuable information and would clearly affect the cartel sustainability.

A Appendix

A.1 *Nash* equilibrium and *Symmetric Nash* equilibrium

In this subsection we give the definition of what a *Nash* equilibrium and a *symmetric Nash* equilibrium of the position auction game are and present the main theoretical prediction of the game.

Definition 3. A strategy profile $\mathbf{b} = (b_1, \cdots, b_k, \cdots, b_m)$ is said to be a Nash equilibrium if there is a compatible assignment ι so that these conditions hold simultaneously:

(i) $\forall k', k \in \mathcal{K}$ with $k' < k$ (Up-Nash):

$$\alpha_k \left(x_{\iota(k)} - b_{\iota(k+1)}\right) \geq \alpha_{k'}(x_{\iota(k)} - b_{\iota(k')})$$

(ii) $\forall k', k \in \mathcal{K}$ with $k' > k$ (Down-Nash):

$$\alpha_k \left(x_{\iota(k)} - b_{\iota(k+1)}\right) \geq \alpha_{k'}(x_{\iota(k)} - b_{\iota(k'+1)})$$

(iii) $\forall k \in \mathcal{K}$:

$$\alpha_k \left(x_{\iota(k)} - b_{\iota(k+1)}\right) \geq 0$$

(iv) $\forall s > m$ and $\forall k \leq m$:

$$\alpha_k \left(x_{\iota(s)} - b_{\iota(k)}\right) \leq 0$$

The first two conditions denote incentives compatibility constraints of the equilibrium bid vector. No player should profitably deviate from the position he is assigned to in equilibrium to any lower or higher positions. The third condition denotes individual rationality constraint: a bidder if assigned cannot profitably deviate to win no position. The last inequality expresses the non-incentives for low-valued players to deviate and win any position.

Note that the shape of prices modifies as upward deviation occurs, which is not in the spirit of usual competitive equilibrium analysis. Players can influence the price they pay at the end and are not taken as given[20]. If bidder $\iota(k)$ decides to undercut the bidder assigned to a position $l < k$ he then expects to pay $b_{\iota(l)}$ and not $b_{\iota(l+1)}$. Hence, the Nash-stability criterion is asymmetric.

We shall give a formal definition of *competitive* equilibria providing minimal conditions taken from Varian (2007), Edelman et al. (2007) under which Börgers et al. (2013) have shown that a *Walrasian* equilibrium is implemented.

Definition 4. A vector of equilibrium bids is said to be *locally envy-free* (LEF) if there exists a compatible assignment ι so that:

$$\alpha_k x_{\iota(k)} - p_k^{gsp} \geq \alpha_{k-1} x_{\iota(k)} - p_{k-1}^{gsp} \ , \ \forall k \in \mathcal{K} \tag{19}$$

with $p_k^{gsp} = \alpha_k b_{\iota(k+1)}$ or equivalently,

$$p_{k-1}^{gsp} - p_k^{gsp} \geq x_{\iota(k)} \left(\alpha_{k-1} - \alpha_k\right) \ , \ \forall k \in \mathcal{K} \tag{20}$$

And is *symmetric* if $\forall k, l \in \mathcal{K}$:

$$\alpha_l (x_{\iota(l)} - p_l^{gsp}) \geq \alpha_k (x_{\iota(l)} - p_k^{gsp}) \tag{21}$$

$$x_{\iota(l)}(\alpha_l - \alpha_k) \geq \alpha_l p_l^{gsp} - \alpha_k p_k^{gsp} \tag{22}$$

with $p_k^{gsp} = b_{k+1}$.

Upward and downward deviations are now symmetric in the price that each bidder expects to pay at the end. The idea is that even if bidders were to swap each other's position, the prices structure should not change. Thus, under the *symmetric* Nash-stability criterion, prices are now taken as given[21]. No player has an incentive to deviate and to bid for the position just above him (that is the *LEF* condition of Edelman et al. (2007)) or more generally to any position above or below him (which is the symmetric condition stated by Varian (2007)). More formally the player assigned to position k has to be *locally indifferent* between winning position $k - 1$, paying his own bid and position k paying the next-highest bid.

Note that the *symmetric* equilibrium conditions described above do not pin down equilibrium bids as it still defined a restricted continuum of bids (or payments) profiles. However, based on the assignment games theory conjunction, the set of *symmetric* Nash equilibria forms what is called a lattice in which one extremum is said to be *buyer-optimal* (eg. Aggarwal, Muthukrishnan and Feldman (2006)) and the other one *seller-optimal*. Without further justifications, Varian (2007), Edelman et al. (2007) have suggested that the lowest outcome would be the most plausible one and have shown that at this state the associated equilibrium profile entails the following result:

[20] Each bidder when contemplating for an upper position does not expect to pay the same price as the one already assigned to that position.

[21] Each player chooses a position that maximized his own payoff taking those prices as given so that there will be strict equality between supply and demand in equilibrium. In this way, the GSP bears also other similarities with uniform-price auctions and the simultaneous ascending price auction described by Milgrom (2000), in which here buyers are single-unit demanders, as the process reach equilibrium prices that clears the market. A result that can also be shown using the procedure based on Hall's theorem of Demange et al. (1986).

Lemma 4. *At the symmetric Nash equilibrium prices under the GSP correspond to VCG payments:*

$$p_i^{gsp}(\mathbf{b}) = \alpha_i b_{i+1} = \sum_{k=i+1}^{m+1} x_k(\alpha_{k-1} - \alpha_k) = p_i^{vcg}(\mathbf{b})$$

hence $\alpha_i b_{i+1} = p_i^{vcg}$ *and given monotone prices* $p_i^{vcg} \geq p_{i+1}^{vcg}$.

Even if truth-telling is not an equilibrium strategy of the GSP auction, the outcome is strictly equivalent to what a VCG mechanism would have implemented. From a revenue perspective, any *symmetric* equilibrium (or *locally envy-free* equilibrium) induces a revenue at least equal to VCG revenues. That is to say, the VCG equilibrium prices constitute the lowest competitive prices induced by the symmetric equilibrium conditions of definition (4) (a result also shown by the works by Feldman et al. (2011), Lucier et al. (2012)).

A.2 Proof of proposition

According to the definition (3) of a *Nash* equilibrium, an equilibrium outcome will be supported by a vector (b_1, b_2, b_3) of equilibrium bids if the following holds:

$$\begin{aligned}
\alpha_1(x_1 - b_2) &\geq \alpha_2(x_1 - b_3) \geq 0 \\
\alpha_2(x_2 - b_3) &\geq \alpha_1(x_2 - b_1) \geq 0 \\
0 &\geq \alpha_2(x_3 - b_2) \\
0 &\geq \alpha_1(x_3 - b_1) \\
b_1 \geq & b_2 \geq b_3 = 0
\end{aligned}$$

rearanging inequalities results in the desired set.

A.3 Proof of proposition (2)

We set the bid of player 1 b_1 arbitrarily high so that no bidder has an incentive to undercut him. It suffices to find a non-negative bid profile (b_2, b_3) so that Nash conditions are satisfied. Then player 1 does not want to move to position 2 if:

$$\begin{aligned}
\alpha_1 (x_1 - b_2) &\geq \alpha_2 (x_1 - b_3) \\
b_2 - \frac{\alpha_2}{\alpha_1} b_3 &\leq x_1 - \frac{\alpha_2}{\alpha_1} x_1
\end{aligned} \tag{23}$$

Player 2 does not want to move to no position if:

$$\begin{aligned}
\alpha_2 (x_2 - b_3) &\geq 0 \\
b_3 &\leq x_2
\end{aligned}$$

Player 3 does not want to move to position 2 if:

$$\alpha_2 (x_3 - b_2) \leq 0$$
$$b_2 \geq x_3$$

And player 1 to move to no position if:

$$\alpha_1 (x_1 - b_2) \geq 0$$
$$b_2 \leq x_1$$

Both b_2 and b_3 are obviously positive thus the left-side of the inequality in (23) is minimized whenever b_2 take its minimal value and b_3 its maximal one. Plugging each value in (23) gives:

$$x_3 - \frac{\alpha_2}{\alpha_1} x_2 \leq x_1 - \frac{\alpha_2}{\alpha_1} x_1$$
$$(\alpha_1 x_1 - \alpha_2 x_1) \geq \alpha_1 x_3 - \alpha_2 x_2$$

A.4 Proof of lemma (1)

This is simple algebraic manipulation. $\underline{b}_2^{\mathcal{N}}$ is strictly positive if:

$$3\alpha_2 x_3 - \alpha_1 b_2^{gsp} - \alpha_2 b_3^{gsp} + \alpha_2 b_3^{\mathcal{N}} > 0$$

multiplying by $\frac{1}{\alpha_1}$ gives

$$\frac{\alpha_2}{\alpha_1} \left(3x_3 + b_3^{\mathcal{N}} - b_3^{gsp}\right) > b_2^{gsp}$$
$$\frac{\alpha_2}{\alpha_1} > \frac{b_2^{gsp}}{3x_3 + b_3^{\mathcal{N}} - b_3^{gsp}}$$

For within-competition to be suppressed we need to have $b_3^{\mathcal{N}} = 0$. As we restrict bids to be non-negative, the lowest-valuing player will bid the maximum value between $x_1 - \theta^{-1} x_1 + \theta^{-1} b_2^{\mathcal{N}} - \frac{1}{3\alpha_2} \Pi_{\mathcal{N}}$ and 0. We can rewrite the first relation into:

$$\begin{aligned}
b_3^{\mathcal{N}} &\geq x_1 - \theta^{-1} x_1 + \theta^{-1} b_2^{\mathcal{N}} - \frac{1}{3\alpha_2} \Pi_{\mathcal{N}} \\
&\geq 2\theta^{-1} b_2^{\mathcal{N}} - \frac{1}{2} \left(\theta^{-1} b_2^{gsp} + b_3^{gsp}\right) + \left(1 - \theta^{-1}\right) \frac{3}{2} x_1 \\
&\geq \frac{1}{2\alpha_2} \left(4\alpha_1 b_2^{\mathcal{N}} - (\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}) - (\alpha_1 - \alpha_2)\, 3x_1\right)
\end{aligned}$$

The right hand side has to be non-negative if the lowest-valuing member was to become an active bidder, thus we have that:

$$4\alpha_1 b_2^{\mathcal{N}} - (\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}) - (\alpha_1 - \alpha_2)\, 3x_1 \geq 0$$

Hence, if the center wants him to refrain from bidding in the target auction it has to set the second-highest valuing member's bid to a value less than:

$$b_2^{\mathcal{N}} \leq (1-\theta)\frac{3}{4}x_1 + \frac{1}{4}\left(b_2^{gsp} + \theta b_3^{gsp}\right)$$

moreover this relation gives the desired relation as regards to θ. The lower boundary is strictly positive if:

$$\begin{aligned}\theta\left(3x_1 - b_3^{gsp}\right) > 3x_1 + b_2^{gsp} - 4b_2^{\mathcal{N}} \\ \theta > \frac{3x_1 - 4b_2^{\mathcal{N}} + b_2^{gsp}}{3x_1 - b_3^{gsp}} = \rho\end{aligned}$$

thus if $\theta < \rho$ then $\min\left\{b_3^{\mathcal{N}}\right\} = 0$.

A.5 Proof of proposition (3)

We want to construct a collusive equilibrium bid profile which is compatible with incentive compatibility constraints defined in (8) and (9) and with the individual rationality constraint defined by the relation (10). We set $\theta = \frac{\alpha_2}{\alpha_1}$.

From player 2's incentive compatibility constraint, he will not deviate for position 1 if the following relation is satisfied:

$$\alpha_2\left(x_2 - b_3^{\mathcal{N}}\right) - \omega_2 + \frac{1}{3}\Pi_{\mathcal{N}} \geq \alpha_1\left(x_2 - b_1^{\mathcal{N}}\right) - \omega_2$$

This relation gives the following conditions:

$$\begin{aligned}b_3^{\mathcal{N}} &\leq x_2 - \theta^{-1}x_2 + \theta^{-1}b_1^{\mathcal{N}} + \frac{1}{3\alpha_2}\Pi_{\mathcal{N}} \\ b_1^{\mathcal{N}} &\geq x_2 - \theta x_2 + \theta b_3^{\mathcal{N}} - \frac{1}{3\alpha_1}\Pi_{\mathcal{N}}\end{aligned}$$

giving the first lower boundary for player 1 and an upper one for player 3.

Now from individual rationality constraint of player 2, he prefers participating in a functioning ring rather than a *non-cooperative* play:

$$\alpha_2\left(x_2 - b_3^{\mathcal{N}}\right) - \omega_2 + \frac{1}{3}\Pi_{\mathcal{N}} \geq \alpha_2\left(x_2 - b_3^{gsp}\right)$$

giving the following relation:

$$\begin{aligned}b_3^{\mathcal{N}} &\leq b_3^{gsp} - \frac{1}{\alpha_2}\left(\omega_2 - \frac{1}{3}\Pi_{\mathcal{N}}\right) \\ &\leq b_3^{gsp} - \frac{1}{3\alpha_2}\left(2\omega_2 - \omega_1\right)\end{aligned}$$

the second upper boundary for player 3.

Now from incentive compatibility constraint of player 1, if he does not want to swap his position for position 2 then it should be the case that:

$$\alpha_1\left(x_1 - b_2^{\mathcal{N}}\right) - \omega_1 + \frac{1}{3}\Pi_{\mathcal{N}} \geq \alpha_2\left(x_1 - b_3^{\mathcal{N}}\right) - \omega_1$$

which gives the following relations:

$$\begin{aligned} b_2^{\mathcal{N}} &\leq x_1 - \theta x_1 + \theta b_3^{\mathcal{N}} + \frac{1}{3\alpha_1}\Pi_{\mathcal{N}} \\ b_3^{\mathcal{N}} &\geq x_1 - \theta^{-1} x_1 + \theta^{-1} b_2^{\mathcal{N}} - \frac{1}{3\alpha_2}\Pi_{\mathcal{N}} \end{aligned}$$

the first upper boundary for player 2 and the lower one for player 3.

Consider now his individual rationality constraint, we get:

$$\alpha_1\left(x_1 - b_2^{\mathcal{N}}\right) - \omega_1 + \frac{1}{3}\Pi_{\mathcal{N}} \geq \alpha_1\left(x_1 - b_2^{gsp}\right)$$

which ends up to the following relation:

$$\begin{aligned} b_2^{\mathcal{N}} &\leq b_2^{gsp} - \frac{1}{\alpha_1}\left(\omega_1 - \frac{1}{3}\Pi_{\mathcal{N}}\right) \\ &\leq b_2^{gsp} - \frac{1}{3\alpha_1}\left(2\omega_1 - \omega_2\right) \end{aligned}$$

giving an upper boundary for player 2.

Finally, looking at incentive compatibility constraint for player 3 we get the following:

$$\begin{aligned} \frac{1}{3}\Pi_{\mathcal{N}} &\geq \alpha_1\left(x_3 - b_1^{\mathcal{N}}\right) \\ \frac{1}{3}\Pi_{\mathcal{N}} &\geq \alpha_2\left(x_3 - b_2^{\mathcal{N}}\right) \end{aligned}$$

giving respectively:

$$\begin{aligned} b_1^{\mathcal{N}} &\geq x_3 - \frac{1}{3\alpha_1}\Pi_{\mathcal{N}} \\ b_2^{\mathcal{N}} &\geq x_3 - \frac{1}{3\alpha_2}\Pi_{\mathcal{N}} \end{aligned}$$

the second lower boundary for player 1 and 2.

And lastly individual rationality constraint ends-up with:

$$\alpha_1 b_2^{\mathcal{N}} + \alpha_2 b_3^{\mathcal{N}} \leq \alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}$$

The above inequalities results in the following equilibrium strategy profile $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$:

Set of collusive bids 1.

$$b_1^{\mathcal{N}} \in \left[\max\left\{x_3 - \frac{1}{3\alpha_1}(\omega_1+\omega_2)\ ;\ x_2 - \theta\left(x_2 - b_3^{\mathcal{N}}\right) - \frac{1}{3\alpha_1}(\omega_1+\omega_2)\right\}\ ;\ \bar{x}\right]$$
$$b_2^{\mathcal{N}} \in \left[\max\left\{0\,;\ \frac{3\alpha_2 x_3 - \alpha_1 b_2^{gsp} - \alpha_2 b_3^{gsp} + \alpha_2 b_3^{\mathcal{N}}}{3\alpha_2 - \alpha_1}\right\}\ ;\ \mathcal{A}\right]$$
$$b_3^{\mathcal{N}} \in \left[\max\left\{0\,;\ \frac{1}{2\alpha_2}\left(4\alpha_1 b_2^{\mathcal{N}} - (\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}) - (\alpha_1 - \alpha_2)\,3x_1\right)\right\}\ ;\ \mathcal{B}\right]$$

with $\theta = \frac{\alpha_2}{\alpha_1}$ *and*

$$\mathcal{A} = \min\left\{\frac{1}{4\alpha_1}\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp} - 2\alpha_2 b_3^{\mathcal{N}} + (\alpha_1-\alpha_2)\,3x_1\right)\ ;\ \frac{1}{\alpha_1}\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp} - \alpha_2 b_3^{\mathcal{N}}\right)\right\}$$
$$\mathcal{B} = \min\left\{b_3^{gsp} - \frac{1}{\alpha_2}\left(\omega_2 - \frac{1}{3}\Pi_{\mathcal{N}}\right)\ ;\ x_2 - \frac{1}{\alpha_2}\left(\omega_2 - \frac{1}{3}\Pi_{\mathcal{N}}\right)\right\}$$

Among these equilibrium bids, an intermediary firm seeking to maximize joint profits reduces to search for the lowest combination of prices that will be paid in the targeting auction.

Lowest collusive bids 1. *Then, given lemma* (1) *reduces the strategy profile to the following:*

if $\theta > \eta$ *then:*

$$b_1^{\mathcal{N}} = x_3 + \frac{(\alpha_1 b_2^{gsp} - \alpha_2 b_3^{gsp}) - \alpha_1 b_2^{\mathcal{N}} - \alpha_2 b_3^{\mathcal{N}}}{3\alpha_1}$$
$$b_2^{\mathcal{N}} = \frac{3\alpha_2 x_3 - (\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp})}{3\alpha_2 - \alpha_1} = x_3 - \gamma$$
$$b_3^{\mathcal{N}} = \max\{0, r\}$$

giving an overall collusive payoff of $\Pi_{\mathcal{N}} = \alpha_1 x_1 + \alpha_2 x_2 - \frac{\alpha_1}{\alpha_1 - 3\alpha_2}\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp} - 3\alpha_2 x_3\right).$

if $\theta \leq \eta$ *then:*

$$b_1^{\mathcal{N}} \in \left[b_2^{\mathcal{N}}, \bar{x}\right]$$
$$b_2^{\mathcal{N}} = b_3^{\mathcal{N}} = \max\{0, r\}$$

and the outcome is no longer efficient.

With $\gamma = \frac{1}{\lambda}\left(\pi_{3,1}^{gsp} - \alpha_2 b_3^{gsp}\right)$ in which $\pi_{3,1}^{gsp} = \alpha_1\left(x_3 - b_2^{gsp}\right)$ the profits of the lowest-valuing member from being assigned to position 1 in a *non-cooperative* play and $\lambda = \alpha_1 - 3\alpha_2$.

A.6 Proof of corollary (2)

To find the optimal bidding functions that maximize the objective function in (6) subject to incentive compatibility constraints defined in (7) (8) (9) and the individual rationality constraint (10) we implement the simplex algorithm.

The proof for the monotonicity and for case (ii) is straightforward.

$\forall \varepsilon \leq \delta^*$ the derivative of $b_2^{\mathcal{N}}$ with respect to ε is given by:

$$\frac{\partial \left(\frac{3\alpha_2 x_3 - (1-\varepsilon)\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}\right)}{3\alpha_2 - (1-\varepsilon)\alpha_1} \right)}{\partial \varepsilon} = \frac{3\alpha_2 \left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp} - \alpha_1 x_3\right)}{\left(3\alpha_2 - (1-\varepsilon)\alpha_1\right)^2} \geq 0 \tag{24}$$

and the derivative with respect to $R^{gsp} = (\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp})$ is given by:

$$\frac{\partial \left(\frac{3\alpha_2 x_3 - (1-\varepsilon) R^{gsp}}{3\alpha_2 - (1-\varepsilon)\alpha_1} \right)}{\partial R^{gsp}} = -\frac{1-\varepsilon}{3\alpha_2 - \alpha_1 (1-\varepsilon)}$$

which is negative if:

$$\begin{aligned} 3\alpha_2 - \alpha_1 (1-\varepsilon) &\geq 0 \\ \frac{\alpha_2}{\alpha_1} &\geq \frac{1-\varepsilon}{3} \end{aligned}$$

which is satisfied by assumption of $\alpha_1 > \alpha_2$.

$$\frac{\partial \left(\frac{3\alpha_2 x_3 - (1-\varepsilon) R^{gsp}}{3\alpha_2 - (1-\varepsilon)\alpha_1} \right)}{\partial R^{gsp}} = -\frac{1-\varepsilon}{\alpha_2 (2+\varepsilon) - 2\alpha_1 (1-\varepsilon)}$$

which is negative.

A.7 Proof of proposition (6)

$\forall \varepsilon \in [0, \delta^*]$ the equilibrium strategy profile is characterized by proposition (5). Thus the center's profit equals the following quantity:

$$\begin{aligned} \Gamma_{\mathcal{N}}^{'} &= \varepsilon \left\{ \alpha_1 \left(b_2^{gsp} - b_2^{\mathcal{N}}\right) + \alpha_2 \left(b_3^{gsp} - b_3^{\mathcal{N}}\right) \right\} \\ &= \varepsilon \left\{ R^{gsp} - \alpha_1 \left(\frac{3\alpha_2 x_3 - (1-\varepsilon)\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}\right)}{3\alpha_2 - (1-\varepsilon)\alpha_1} \right) \right\} \end{aligned}$$

and $\forall \varepsilon \in (\delta^*, 1)$ it takes the following quantity:

$$\begin{aligned} \Gamma_{\mathcal{N}}^{''} = \varepsilon \Bigg\{ R^{gsp} &- \alpha_1 \left(\frac{(2+\varepsilon)\alpha_2 x_3 - (1-\varepsilon)\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}\right) - (1-\varepsilon)(\alpha_1 - \alpha_2) x_1}{(2+\varepsilon)\alpha_2 - (1-\varepsilon) 2\alpha_1} \right) \\ &- \alpha_2 \left(\frac{x_1 (\alpha_1 - \alpha_2)\left((1-\varepsilon)\alpha_1 - 3\alpha_2\right) - (1-\varepsilon)(\alpha_1 + \alpha_2)\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp}\right) + (4-\varepsilon)\alpha_1 \alpha_2 x_3}{\alpha_2 \left((2+\varepsilon)\alpha_2 - (1-\varepsilon) 2\alpha_1\right)} \right) \Bigg\} \end{aligned}$$

The first and second derivative of $\Gamma_{\mathcal{N}}^{'}$ are respectively equal to:

$$\frac{\partial}{\partial \varepsilon}\left(\Gamma_{\mathcal{N}}^{'}\right) = 3\alpha_2 (3\alpha_2 - \alpha_1) \frac{\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp} - \alpha_1 x_3\right)}{\left(3\alpha_2 - (1-\varepsilon)\alpha_1\right)^2}$$

which is positive if $3\alpha_2 \geq \alpha_1$ and

$$\frac{\partial^2}{\partial^2 \varepsilon}\left(\Gamma'_{\mathcal{N}}\right) = -6\alpha_1\alpha_2\left(3\alpha_2 - \alpha_1\right)\frac{\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp} - \alpha_1 x_3\right)}{\left(3\alpha_2 - \left(1-\varepsilon\right)\alpha_1\right)^3}$$

which is of negative sign whenever $3\alpha_2 \geq \alpha_1$. Thus on the domain $[0, \delta^*)$ the intermediary firm profit function is concave. The function is non-differentiable in $\varepsilon = \delta^*$ and the first and second derivative of $\Gamma''_{\mathcal{N}}$ are respectively given by:

$$\frac{\partial}{\partial \varepsilon}\left(\Gamma''_{\mathcal{N}}\right) = -6\alpha_2\left(\alpha_1 - \alpha_2\right)\frac{\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp} + x_1\left(\alpha_1 - \alpha_2\right) - 2\alpha_1 x_3\right)}{\left(\left(2+\varepsilon\right)\alpha_2 - \left(1-\varepsilon\right)2\alpha_1\right)^2}$$

which is of negative sign and

$$\frac{\partial^2}{\partial^2 \varepsilon}\left(\Gamma''_{\mathcal{N}}\right) = 12\alpha_2\left(\alpha_1 - \alpha_2\right)\left(2\alpha_1 + \alpha_2\right)\frac{\left(\alpha_1 b_2^{gsp} + \alpha_2 b_3^{gsp} + x_1\left(\alpha_1 - \alpha_2\right) - 2\alpha_1 x_3\right)}{\left(\left(2+\varepsilon\right)\alpha_2 - \left(1-\varepsilon\right)2\alpha_1\right)^2}$$

which is of positive sign. Thus, the profit function is a convex function over the domain $\left(\delta^*, 1\right)$.

A.8 Proof of proposition (7)

The proof is straightforward. Take the limit as ε grows to 1 we see that collusive payoffs converge to the *non-cooperative* level we get:

$$\lim_{\varepsilon \mapsto +\infty}\left\{\alpha_k\left(x_k - b_{k+1}^{\mathcal{N}}\right) - \left(\alpha_k b_{k+1} - \alpha_k b_{k+1}^{\mathcal{N}}\right) + \frac{1-\varepsilon}{n}\sum_{j=2}^{m}\alpha_j b_{j+1}^{gsp} - \frac{1-\varepsilon}{n}\sum_{j=2}^{m}\alpha_j b_{j+1}^{\mathcal{N}}\right\} \tag{25}$$

which leads to $\alpha_k\left(x_k - b_{k+1}^{gsp}\right) = \pi_k^{gsp}$. The objective of the center is thus now to maximize the function $SP = \sum_{k=1}^{n}\left(\alpha_k\left(x_k - b_{k+1}^{gsp}\right)\right)$ under the same *IC* constraints of the *non-cooperative* equilibrium given by conditions of definition (3) which entails $\mathbf{b}_L^{gsp}$ as a natural equilibrium outcome. One can also set $\varepsilon = 1$ into the equilibrium collusive bidding functions defined in corollary (3) to get:

$$\begin{aligned} b_2^{\mathcal{N}} &= x_3 = b_2^{gsp,low} \\ b_3^{\mathcal{N}} &= x_1 - \frac{\alpha_1}{\alpha_2}\left(x_1 - x_3\right) = b_3^{gsp,low} \end{aligned}$$

which is the desired outcome.

B Figures and Tables

We run 1000 instance of the one-shot *GSP* game in which following Cary et al. (2008), each valuation is drawn from a distribution $G(x) \sim \mathcal{N}(500, 200)$ setting the value of X_1, X_2 and X_3 respectively to $x_1 = 592.7$, $x_2 = 565.535$ and $x_3 = 437.331$. On average, we observe a *ctr* of 0.23%

on higher positions which allows us to reasonably set $\lambda = 0.23$ giving $E(\alpha) = 4.3$ (Synodiance 2013 synodiance.ctr.study2013) thus $H(\alpha) \sim \mathcal{E}_{(0.23)}$. The exponential law will generate numbers lying between 0 and 1 which can be straightly interpreted as clicks probability or clicks rates.

B.1 Bids, profits and revenues

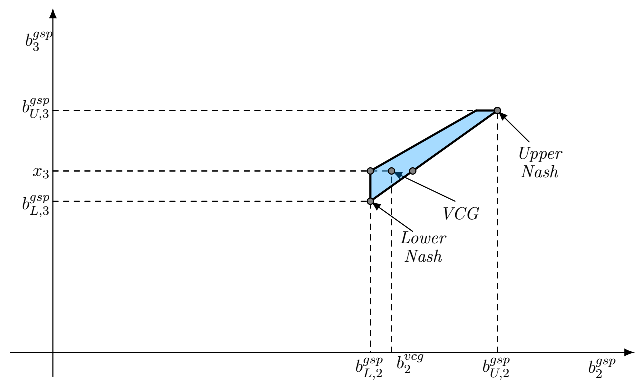


Figure 4: Set of Nash bids with the highest-valuing player's bid set arbitrary high

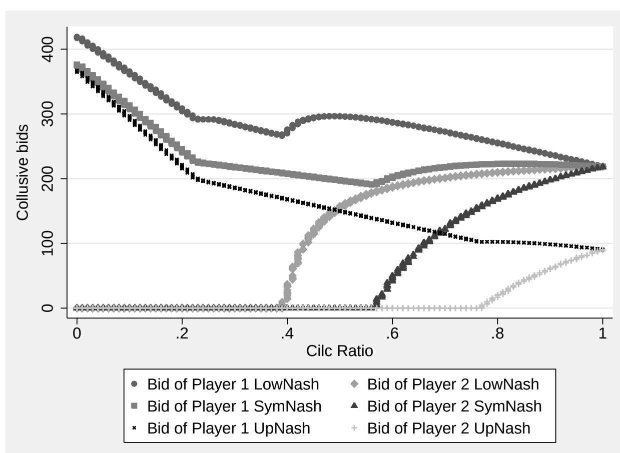
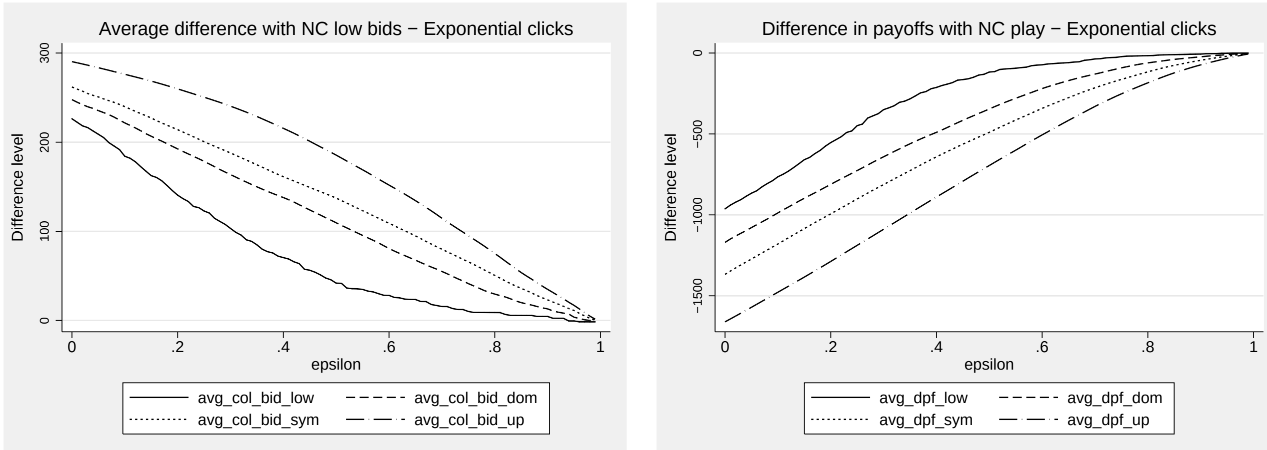


Figure 5: Equilibrium collusive bids when $\varepsilon = 0$ against θ respectively for $\mathbf{b}_L^{\mathcal{N}}, \mathbf{b}_{Sym}^{\mathcal{N}}, \mathbf{b}_U^{\mathcal{N}}$ when $H(\alpha) \sim \mathcal{E}_{(0,23)}$.



(a) Difference in bids (b) Difference in payoffs

Figure 6: Average difference with the bids sustaining $\mathbf{b}_L^{gsp}$ and payoffs against each $\mathbf{b}^{gsp}$ when the center set a fee $\varepsilon \geq 0$.

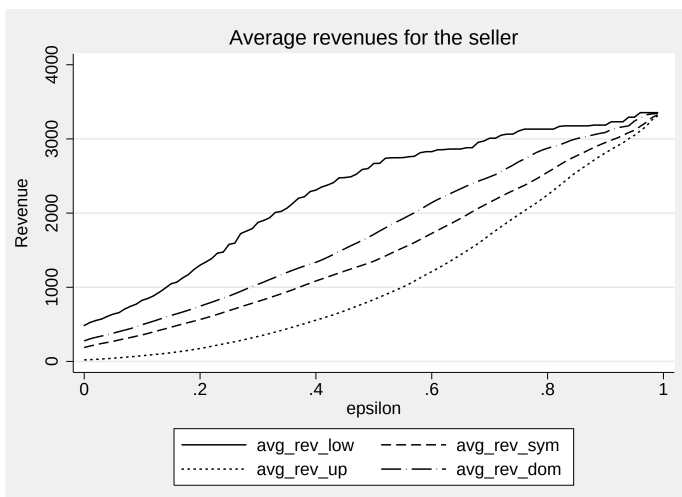


Figure 7: Average seller's revenue according to $\mathbf{b}_L^{gsp}, \mathbf{b}_{dom}^{gsp}, \mathbf{b}_{sym}^{gsp}$ and $\mathbf{b}_U^{gsp}$ when the center set a fee $\varepsilon \geq 0$.

B.2 Comparison between *non-cooperative* outcome and collusive outcome and thresholds for ε

	Benevolent Center									
	Non-Cooperative Play					Cartel				
	$\mathbf{b}_L^{gsp}$	$\mathbf{b}_{Dom}^{gsp}$	$\mathbf{b}_{Sym}^{gsp}$	$\mathbf{b}_U^{gsp}$	Average	$\mathbf{b}_L^{gsp}$	$\mathbf{b}_{Dom}^{gsp}$	$\mathbf{b}_{Sym}^{gsp}$	$\mathbf{b}_U^{gsp}$	Average
Surplus										
Player 1	1024.32	1024.32	466.81	60.98	644.11	2039.84	2231.134	1878.852	1756.32	1976.53
Player 2	700.97	287.65	287.65	0	319.06	1716	1494.49	1699.67	1695.34	1651.52
Player 3	.	.	.	.	.	1015.54	1206.84	1412.02	1695.34	1332.43
Contributions										
Player 1	.	.	.	.	.	2478.76	2639.27	3254.81	3817.13	3047.5
Player 2	.	.	.	.	.	567.86	981.24	981.24	1268.9	949.8
Player 3	.	.	.	.	.	0	0	0	0	
Average Contributions	.	.	.	.	.	1015.542	1206.84	1412.02	1332.43	
Seller Revenue	3450.82	3864.2	4421.7	5115.17	4212.9	404.2	243.7	185.64	29.15	215.67
Total Surplus	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15
Cartel Surplus	.	.	.	.	.	4771.95	4932.46	4990.5	5147	4960.48
Bids										
Player 1	453.53	514.83	592.71	592.71	538.44	304.38	274.31	244.81	194.88	254.6
Player 2	437.3	437.3	514.83	581.96	492.86	78.89	49.9	38.98	7.04	43.73
Player 3	170.14	437.3	437.3	565.54	402.59	0	0	0	0	0
Average Bids	353.67	463.16	514.96	580.07		127.76	108.1	94.6	67.31	

Table 1: Equilibrium outcomes with a benevolent center for $H(\alpha) \sim \mathcal{E}_{(0.23)}$ and $G(x) \sim \mathcal{N}_{(500,200)}$

	Corresponding NC outcomes				
	$\mathbf{b}_L^{gsp}$	$\mathbf{b}_{Dom}^{gsp}$	$\mathbf{b}_{Sym}^{gsp}$	$\mathbf{b}_U^{gsp}$	Average
Surplus					
Player 1	790.28	790.28	633.53	104.9	579.77
Player 2	685.22	495.27	495.27	0	418.94
Player 3	.	.	.	.	.
Seller Revenue	3723.88	3913.85	4070.6	5094.43	4200.69
Total Surplus	5199.3	5199.3	5199.3	5199.3	5199.3
Bids					
Player 1	431.12	466.65	592.71	592.71	520.79
Player 2	437.3	437.3	466.65	571.75	478.25
Player 3	385.58	437.3	437.3	565.54	456.45
Average Bids	418	447.1	498.9	576.67	

Table 3: Equilibrium *NC* outcomes corresponding to the non-neutral *center* case for $H(\alpha) \sim \mathcal{E}_{(0.23)}$ and $G(x) \sim \mathcal{N}_{(500,200)}$

	Active Center														
	ε'					$\varepsilon = \delta^*$					ε''				
	$\mathbf{b}_L^{gsp}$	$\mathbf{b}_{Dom}^{gsp}$	$\mathbf{b}_{Sym}^{gsp}$	$\mathbf{b}_U^{gsp}$	Mean	$\mathbf{b}_L^{gsp}$	$\mathbf{b}_{Dom}^{gsp}$	$\mathbf{b}_{Sym}^{gsp}$	$\mathbf{b}_U^{gsp}$	Mean	$\mathbf{b}_L^{gsp}$	$\mathbf{b}_{Dom}^{gsp}$	$\mathbf{b}_{Sym}^{gsp}$	$\mathbf{b}_U^{gsp}$	Mean
Threshold	0.085	0.115	0.136	0.236	.	0.169	0.23	0.271	0.472	.	0.254	0.345	0.407	0.709	.
Surplus															
Player 1	1554.94	1604.94	1487.98	1203.46	1462.83	1448.38	1448.38	1291.63	763.1	1237.87	790.272	984.68	867.175	324.98	741.77
Player 2	1449.94	1309.97	1349.76	1098.47	1302.03	1343.4	1153.41	1153.41	658.10	1077.07	685.28	689.71	728.85	219.98	580.98
Player 3	764.66	814.67	854.46	1098.47	883.06	658.1	658.1	658.1	658.1	658.1	0	194.41	233.65	219.98	162.01
Center Payoff	201.31	314.73	410.47	1025.63	488.03	366.8	556.77	713.52	1737.35	843.61	0	287.91	458.4	1455.03	550.33
Seller Revenue	1228.58	1155.12	1096.76	773.41	1063.47	1382.78	1382.78	1382.78	1382.78	1382.78	3723.9	3042.72	2911.25	2979.45	3164.33
Total Surplus	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43
Cartel Surplus	3769.54	3729.58	3692.2	3400.4	3647.93	3449.86	3259.88	3103.14	2079.31	2973.05	1475.55	1868.8	1829.78	764.95	1484.8
Contributions															
Player 1	995.7	1069.15	1284.28	2136.15	1371.34	841.5	841.5	998.25	1526.78	1052	0	243.38	446.24	947.5	409.23
Player 2	1499.6	1689.58	1689.58	2184.9	1765.9	1499.6	1689.58	1689.58	2184.9	1765.9	0	627.76	713.1	1167.49	627.1
Player 3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
Average	831.77	919.58	991.28	1440.34	.	780.37	843.69	895.94	1237.22	.	0	290.38	386.45	705	.
Bids															
Player 1	283.71	274.37	266.93	218.38	260.85	303.73	303.73	303.73	303.73	303.73	437.3	397.39	389.06	390.81	403.65
Player 2	240.38	226.73	215.9	152.52	208.88	269.17	269.17	269.17	269.17	269.17	437.3	388.1	378.43	382.65	396.63
Player 3	.	.	.	.	.	.	.	.	.	.	385.58	274.14	253.07	265.43	294.56
Average Bids	174.7	167.03	160.94	123.63	.	190.97	190.97	190.97	190.97	190.97	420.1	353.21	340.18	346.3	.

Table 2: Collusive outcomes when the center set its share to $\varepsilon = \{0.5\delta^*, \delta^*, 1.5\delta^*\}$ for $H(\alpha) \sim \mathcal{E}_{(0.23)}$ and $G(x) \sim \mathcal{N}_{(500,200)}$

	Threshold δ^*			
	b_L^{gsp}	b_{dom}^{gsp}	b_{U-dom}^{gsp}	b_U^{gsp}
δ^*	0.143	0.241	0.318	0.516

Table 4: Threshold δ^* values according to $\mathbf{b}_L^{gsp}, \mathbf{b}_{dom}^{gsp}, \mathbf{b}_{U-dom}^{gsp}, \mathbf{b}_U^{gsp}$

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