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The Impact of Talents and Preferences on Income Inequality

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L'Impact des Talents et des Préférences sur l'Inégalité des Revenus

Résumé

Les répartitions des revenus que l'on peut observer tendent à varier, tant entre pays, qu'au cours du temps pour le même pays. La question qui nous intéresse est de savoir d'où viennent ces inégalités et qu'est-ce qui explique leurs différences. En limitant notre attention à une économie d'artisans sans imposition et où les individus ont des préférences identiques mais des productivités différentes, nous étudions l'impact sur l'inégalité des revenus du travail de certaines modifications de la distribution des productivités. En supposant ensuite que la distribution des productivités est fixée, nous cherchons quelles sont les modifications des préférences qui conduisent à une répartition plus égale des revenus. Enfin, nous examinons la question de savoir comment des changements simultanés des préférences et de la distribution des productivités interagissent dans la formation de la distribution des revenus du travail.

Mots-clés : Revenu, Inégalités, Productivités, Préférences, Dominance au sens de Lorenz.

The Impact of Talents and Preferences on Income Inequality

Abstract

Different trends in the distribution of income across countries and in the same country over time are typically observed. The general question we are interested in is to know where these inequalities come from and what explains their differences. By restricting our attention to an artisan economy with no taxation and where individuals have identical preferences but different productivities, we study the impact on the inequality of labour income of particular changes in the way productivities are allocated. Then, assuming that the distribution of productivities is fixed, we look for the modifications of the preferences that lead to a more even distribution of income. Finally, we examine the question of how simultaneous changes in preferences and in the distribution of productivities interact in shaping the distribution of labour income.

Keywords: Income, Inequality, Productivities, Preferences, Lorenz Dominance.

JEL: D31, D63.

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1. Introductory Remarks[1]

1.1. Motivation

In this paper, we are interested in the way preferences and productivities interact in the determination of the *distribution of income* among the society's members. There is ample empirical evidence that *income inequality* has continuously risen during the last thirty years in most developed and emerging countries. For instance, the OECD reports that the Gini index in developed countries has increased by about seven percent on average, reaching unprecedented levels in countries like Mexico, Chile or Turkey, and to a lesser extent, in the US (see OECD (2008)).[2] Since for many persons or households *labour income* is by far the most important income component, the story behind this rising inequality is one about changes in earnings. Indeed, while it has fallen steadily over the last thirty years, the share of labour income still represents sixty to seventy percent of total household income according to OECD (2012, Chapter 3). At the same time, the share of income going to highly paid workers has increased in many cases, whereas the share going to low-paid workers has declined.

These observations have given rise to numerous studies mainly by labour economists – starting in the mid-1990s – that have concentrated mostly on the examination of the pattern of the *distribution of wages* and on the factors that are likely to shape it (see, e.g., Bryan and Martinez (2008) for a survey). Variations in hourly wages were generally considered the major contributory factor in levels of gross earnings inequality among all workers in most OECD countries.[3] For instance, using the decomposition of the variance of the logarithms, Blau and Kahn (2011) find that wage rate variation explains on average fifty five percent of gross earnings variation across the OECD countries. At the same time, inequality in annual earnings tends to be higher than hourly wage inequality in most of the OECD countries. Therefore, while variations in hourly wage rates appear to be the largest contributory factor in gross earnings inequality among all workers in most countries, changes in earnings inequality over time seem to be driven as much by trends in hours worked. As regards the latter, one observes a growing divide between higher-wage earners and lower-wage earners in many OECD countries. The annual hours worked declined among the lower-wage group, sometimes significantly, while at

[1] This paper forms part of the research project **Heterogeneity and Well-Being Inequality** (Contract No. HEWI/ANR-07-FRAL-020) of the ANR-DFG programme whose financial support is gratefully acknowledged. A shorter version will appear in a forthcoming issue of *Social Choice and Welfare* and we take this opportunity to thank an associate editor and two anonymous referees for very useful comments and suggestions. The second author would like to thank Stephen Bazen for the time spent at discussing the issues addressed in the paper. Needless to say, none of the persons mentioned above should be held responsible for remaining deficiencies.

[2] The Gini index is known to be more sensitive to income changes at the middle than at the tails of the distribution (see, e.g., Atkinson (1970, Section 4)). Since there is evidence that the income gaps have mostly increased among the richest and the poorest households, one may reasonably expect that the figures reported in the OECD report underestimate the changes in inequality during the last three decades.

[3] Although we do not follow this route, it must be emphasised that focusing on the distribution of hourly wages has its own merits from the equality of opportunity point of view. Indeed, because the wage rate essentially captures the size of the budget set from which the agent optimally chooses her bundle, it can be taken as a measure of the size of the agent's opportunity set (see Fleurbaey and Maniquet (2015)).

the same time higher-income workers tend to work more than lower-income workers. These observations seem to indicate that variations in hours worked play a non-negligible role in portraying the pathway between wage inequality and annual earnings inequality.

1.2. The Approach Followed in the Paper

While complex models where supply and demand and institutional arrangements interplay in the explanation of the observed inequality trend of incomes during the three last decades, the short review of the literature above suggests that the dispersion of wages and of hours worked are important determinants of earnings inequalities. In this paper we would like to investigate how productivities and preferences interact in the determination of the distribution of income between the society's members in a purely abstract model. Thus, our approach is far less ambitious than the investigations pursued by – among others – labour economists, and our results are likely to be modest in terms of applicability to real world problems. We nevertheless are confident that the paper provides some useful small steps in order to better comprehend how personal traits like talents and tastes contribute to shaping income inequalities.

To this aim we consider an *artisan economy* with two commodities: consumption and leisure. Each individual is completely identified by a preference ordering in the leisure-consumption space and her productivity measuring her contribution to total output. We assume constant returns to scale so that the individual's labour income – or, equivalently, gross income or earnings – is equal to her working time multiplied by her productivity. Because in such an economy wages are equal to productivities, the model is expected to draw links between the measurement of inequality in wages addressed by labour economists and the more general question of inequality in earnings. There is no taxation in our model and consumption is therefore equal to (labour) income. For simplicity – and in order to make things tractable – we assume throughout that all members of the same society have identical preferences. On the other hand, the members of a given society differ in terms of their productivities, which reflects the fact that they have different talents. A society is then completely characterised by a preference ordering defined on the labour-consumption space and the distribution of productivities among its members.

Labour economists traditionally appeal to the decomposition of the variance of the logarithms in order to measure the respective contribution of wages and hours worked to the inequalities of earnings. One problem with this decomposition procedure is that the number of hours worked depend on the preferences but also on the wage rates: thus, the two variables entering the decomposition are not independent. Furthermore, even though the decomposition of the variance of the logarithms tells us that a non-negligible part of earnings inequalities comes from inequalities in the number of hours worked, it does not say anything about the origins of these observed differences in labour supplies. We follow a different route and adopt a comparative static framework in order to investigate the implications of *particular changes* in individuals' productivities for the distribution of consumption – or, equivalently, earnings in our model – among individuals. This means that we do not consider all possible modifications but rather focus on those that we believe are the most important and at the same time allow us to derive unambiguous conclusions regarding the direction of the inequality changes. More precisely, we address the following series of questions:

(a) Is it always the case that less dispersed productivities among the population give rise to lower income inequality?

(b) If it is not the case, then is it possible to identify the properties of the utility functions – and beyond that the preferences – that guarantee that income inequality decreases when productivities become more concentrated?

(c) Assuming that individuals' productivities are given, which modifications of the preferences would lead to a more equal distribution of income between individuals?

(d) How do simultaneous modifications of the distribution of productivities and of the preferences impact the distribution of income?

Most studies in the labour literature rely on specific indices like the Gini coefficient or the variance of the logarithms when comparing distributions of income within and across societies.[4] Rather than appealing to particular inequality indices in order to compare distributions of income within and across societies, we prefer the approach in term of dominance. We are indeed searching for robust results and the dominance approach avoids much of the arbitrariness inherited from the choice of a specific inequality index. Admittedly, this has a cost: it is not always possible to decide whether one distribution is more or less unequal than another. We adopt here the standard approach which consists in appealing to the relative and absolute *Lorenz quasi-orderings* for comparing distributions of income (see Kolm (1969), Atkinson (1970), Sen (1997), Shorrocks (1983), or Moyes (1999) among others).

1.3. Organisation of the Paper

We present in Section 2 the model that we will use throughout the document. It is a very simplified economy in which each agent separately decides the amount of time she devotes to work, which, given her productivity, determines her labour income. No trade is allowed and, as a result, an agent can only consume what she produces. Since there is no taxation, consumption equals labour income, that is the number of hours worked multiplied by productivity plus eventually an exogenous income. Section 3 is devoted to the presentation of the criteria we use for appraising changes in the allocation of individual productivities. We make a distinction between those modifications of the distribution of productivities that reduce dispersion and those that improve efficiency. Section 4 contains our main results concerned with the identification of the properties of the consumption function that ensure that consumption inequality decreases as a result of less dispersed productivities. We adopt here the standard practice that consists in using the relative Lorenz criterion for making inequality comparisons. The elasticity of the consumption function is shown to be the key variable for determining the impact on income inequality of less dispersed productivities. The relative inequality approach has been challenged by some authors (see in particular Kolm (1976)) and alternatives to the relative Lorenz quasi-ordering have been proposed. We provide in Section 5 the analogues of

[4] Presumably, the variance of the logarithms is extensively used by labour economists because of it decomposability properties allowing one to measure the respective contribution to overall earnings inequality of wages and hours worked.

the characterisation results of the previous section when consumption inequality is evaluated by means of the absolute Lorenz quasi-ordering. We identify in Section 6 the properties of the utility functions that constitute the counterparts of the consumption elasticities conditions in the particular case where preferences are quasi-linear. Section 7 concludes the paper summarising our main findings, pointing at limitations and suggesting avenues for further research. Finally, Section 8 contains the proofs of our main results while Section A provides additional technical material.

2. Notation and Preliminary Definitions

2.1. The Stylised Economy

We consider an *artisan economy* with n individuals ($n \geqslant 2$) and two commodities: *consumption* c and *labour time* ℓ with $0 \leqslant \ell \leqslant T$, where T represents the maximum amount of leisure available. All the individuals belonging to the same society have identical preferences over the labour-consumption space represented by an ordinal (common) *utility function* $u(c, \ell)$ which is assumed to be (i) twice differentiable, (ii) increasing in consumption and decreasing in labour time, and (iii) strictly quasi-concave with respect to the consumption-labour time bundle. *Gross income* z is determined by *productivity* $w \in (0, \overline{w})$ and labour time ℓ according to the constant returns to scale technology $z = w\,\ell$.[5] Upon substitution into the direct utility function, we obtain the *personalised utility function* $U(c,z,w) := u(c, z/w)$ that depends on the agent's productivity. It follows from the properties of the direct utility function that $U(c,z,w)$ is (i) twice differentiable, (ii) increasing in consumption, and (iii) strictly quasi-concave with respect to the consumption-gross income bundle. As it is common practice in the literature, we further impose that $U(c,z,w)$ verifies the Spence-Mirrlees condition

$$MRS(c,z,w) := -\frac{U_z(c,z,w)}{U_c(c,z,w)} \text{ is decreasing in } w,\ \forall\,(c,z) \gg (0,0), \tag{2.1}$$

according to which, the slopes of the personalised indifference curves decrease with productivity at any given consumption-gross income bundle.[6] We find it convenient for later use to indicate by $\mathscr{U}$ the set of utility functions such that the above properties are satisfied and we further note that $\mathscr{U}$ is closed under twice differentiable and increasing transformations.

2.2. The Market Equilibria

The optimisation program of an individual endowed with productivity $w \in (0, \overline{w})$ and the utility function $u \in \mathscr{U}$ writes:

$$(c,z) \max U(c,z,w) \text{ s.t. } 0 < c \leqslant z \text{ and } \frac{z}{w} \leqslant T. \tag{2.2}$$

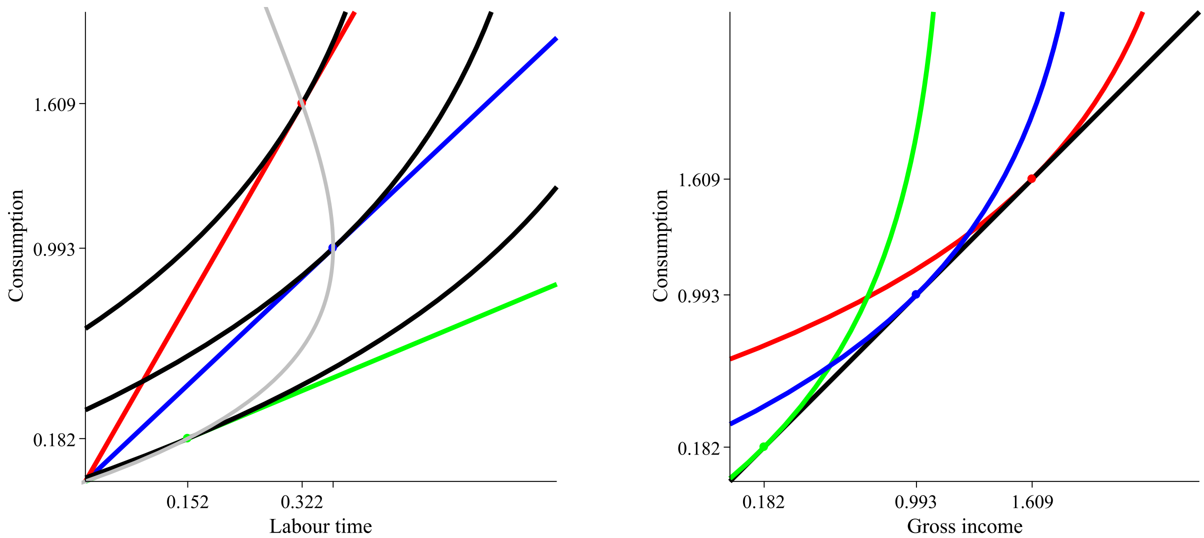
[5] The assumption that productivities are upper bounded plays no role in the proofs of all our results with the exception of one of them where we have not been able to find a way of reasoning that does not use it. Nevertheless, this assumption is not restrictive to the extent that the upper limit $\overline{w}$ can be arbitrarily large.

[6] Equivalently, in the labour-consumption space, the slopes of the indifference curves decrease with labour time for any given consumption level.the slopes of the personalised indifference curves decrease with gross income for any given consumption level.

Given the utility function $u \in \mathscr{U}$, we indicate by $W(u) \subseteq (0, \overline{w})$ the range of productivities such that the optimisation problem above has a *unique interior* solution. We denote by $C(w;u)$ and $Z(w;u)$ the solution of problem (2.2): the consumption and labour income of an individual depend on her productivity w and her preferences represented by the utility function u. It follows from our assumptions that the consumption function $C(\cdot\,;u)$ and the gross income function $Z(\cdot\,;u)$ are increasing with productivity (see, e.g., Ebert and Moyes (2007, Lemma 1)). We also note that, since utility is increasing in consumption and since there is no taxation, we necessarily have $C(w;u) = Z(w;u)$, for all $w \in W(u)$.[7] The Spence-Mirrlees condition guarantees that gross income – and therefore consumption – increases with productivity (see, e.g., Myles (1995, Theorem 5.1)). One can immediately derive the labour supply function $L(\cdot\,;u)$ defined by $L(w;u) = Z(w;u)/w$, which, contrary to gross income, is not necessarily monotonic with respect to productivity.

While the members of the same society have similar tastes and values, they may differ in their productivities. A typical allocation of productivities is a vector $\mathbf{w} := (w_1, \ldots, w_n)$ such that $0 < w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$ and we denote by $\mathscr{W}$ the set of such allocations. Given $u \in \mathscr{U}$, we indicate by $\mathscr{W}(u)$ the set of productivity allocations $\mathbf{w} := (w_1, \ldots, w_n) \in \mathscr{W}$ such that $w_i \in W(u)$, for all $i = 1, 2, \ldots, n$, and by $C(\mathbf{w};u) := (C(w_1;u), \ldots, C(w_n;u))$ the corresponding distribution of the agents' consumptions at the market equilibrium. We have represented

Figure 2.1: Equilibria in the labour-consumption and gross income-consumption spaces



in Figure 2.1 the equilibrium allocations for three different levels of productivity but the same preferences in the labour-consumption space and in the gross income-consumption space, respectively. We can see that for these preferences, which can be represented by the utility function $u^{(6)}(c,\ell) = -e^{-c} - \ell$, labour supply does not increase monotonically with productivity in contrast to what happens for gross income and consumption.

[7] We note that $C(\cdot\,;\tilde{u}) = C(\cdot\,;\hat{u})$ and $Z(\cdot\,;\tilde{u}) = Z(\cdot\,;\hat{u})$, whenever $\tilde{u} = \varphi \circ \hat{u}$ and φ is increasing. The utility functions $\tilde{u}$ and $\hat{u}$ represent distinct preferences if there exists φ increasing and $(c,\ell) \in \mathbb{R}_{++} \times (0,T]$ such that $\tilde{u}(c,\ell) \neq \hat{u}(c,\ell)$.

3. Appraising Modifications in the Distribution of Productivities

Other things equal, it is clear that, if all agents in the economy have the same productivity, then there is no room for income inequality. It is the fact that agents differ in their productivities that explains why inequalities in income arise in our stylised world. One may conceive of different types of modifications of the allocation of productivities that affect the distribution of income. Here we focus on two particular aspects of the distribution of productivities that might contribute to explain the shape of the distribution of income: *concentration* and *efficiency*.

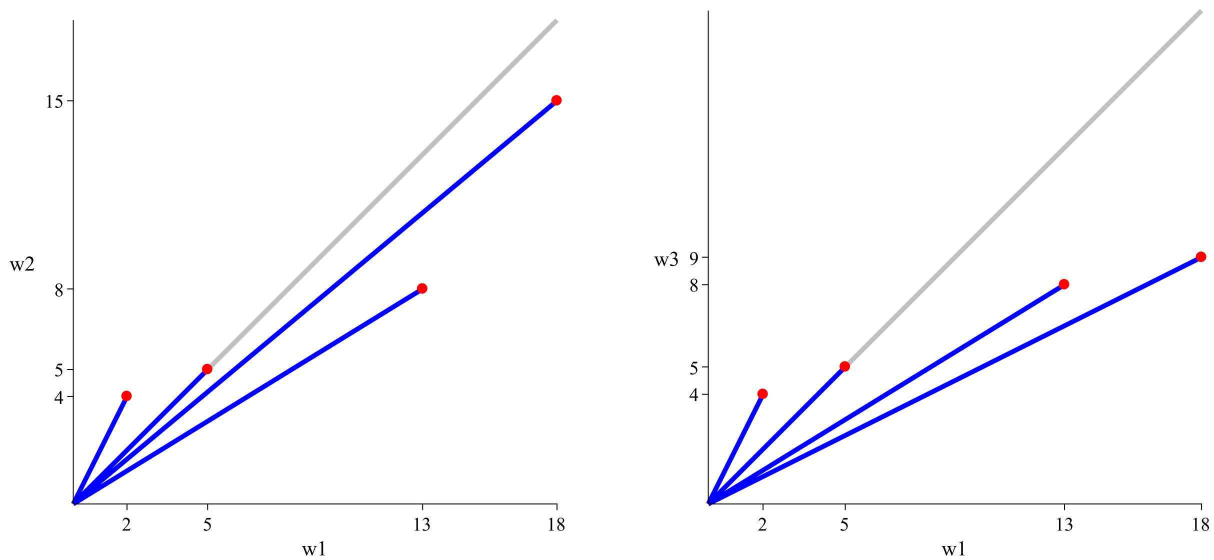
In order to measure the concentration of productivities, we appeal to the *relative differentials* quasi-ordering repeatedly used in the income taxation literature (see, e.g., Moyes (1994), Le Breton, Moyes, and Trannoy (1996) among others).[8]

Definition 3.1. Given two distributions of productivity $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}$, we say that $\mathbf{w}^*$ *dominates* $\mathbf{w}^\circ$ *in relative differentials*, which we write $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$, if and only if:[9]

$$\frac{w_i^*}{w_i^\circ} \geqslant \frac{w_j^*}{w_j^\circ},\ \forall i = 1, 2, \ldots, j-1,\ \forall j = 2, 3, \ldots, n. \tag{3.1}$$

The preceding definition is illustrated in Figure 3.1 where we compare distributions $\mathbf{w}^1 = (2, 5, 13, 18)$, $\mathbf{w}^2 = (4, 5, 8, 15)$, and $\mathbf{w}^3 = (4, 5, 8, 9)$. The figure confirms the incompleteness of the relative differentials quasi-ordering: $\mathbf{w}^3 \geq_{RD} \mathbf{w}^1$ but $\mathbf{w}^1$ and $\mathbf{w}^2$ cannot be ranked. Certainly, the relative differentials quasi-ordering is very demanding to the extent that it re-

Figure 3.1: Relative differentials ranking of distributions $\mathbf{w}^1$, $\mathbf{w}^2$ and $\mathbf{w}^3$



quires in particular that the ratios of productivities computed over all percentiles are reduced for one distribution of productivities to dominate another. On the other hand, most of the

8 This criterion – also known as the *ratio dominance* quasi-ordering (see Preston (1990)) is related to what is referred to as the *dispersive quasi-ordering* in the statistical literature (see, e.g., Shaked and Shanthikumar (1994, Section 2.B) and the references therein).

9 Given a binary relation $\geq_J$ over a set $\mathscr{S} \subseteq \mathbb{R}^n$ ($n \geqslant 2$), we define in the usual way its asymmetric and symmetric components, which we indicate by $>_J$ and $\sim_J$, respectively.

measures currently used to appreciate the dispersion of wages in the literature like the Gini coefficient or the interquantile ratios are consistent with the relative differentials quasi-ordering. For instance, $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$ guarantees that an index like the ratio of the ninth decile over the first decile – repeatedly used for assessing the inequality of wages (see, e.g., OECD (2011)) – will record that $\mathbf{w}^*$ is less dispersed than $\mathbf{w}^\circ$.

For later use, we denote by $\nu(\mathbf{w})$ the geometric mean of the distribution of productivities $\mathbf{w} := (w_1, w_2, \ldots, w_n)$. It is worth noting that the relative differentials quasi-ordering is closely related to following type of transformation that constitutes a variant of the notion of a progressive transfer used in the inequality literature.

Definition 3.2. Given two distributions of productivity $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}$, we say that $\mathbf{w}^*$ is obtained from $\mathbf{w}^\circ$ by means of a *uniform proportional progressive transfer* if there exists $\lambda, \xi > 1$ and two individuals i, j $(1 \leqslant i < j \leqslant n)$ such that:

(3.2a) $$w_h^* = \lambda\, w_h^\circ, \forall h \in \{1, 2, \ldots, i\};\; w_h^* = w_h^\circ/\xi, \forall h \in \{j, j+1 \ldots, n\};$$

(3.2b) $$w_h^\circ = w_h^*, \forall\, h \in \{i+1 \ldots, j-1\};\; \text{and}$$

(3.2c) $$\lambda^i = \xi^{n-j+1}.$$

The notion of a uniform proportional progressive transfer is reminiscent of – but distinct from – the concept of a *proportional transfer* introduced by Fleurbaey and Michel (2001) in the context of intergenerational inequality and optimal growth. Both types of transfers require that proportional amounts of the attribute are taken from richer individuals to be given to poorer ones and that the relative positions of all individuals are preserved. The difference is that a uniform proportional progressive transfer imposes in addition that, if some fraction of the attribute is taken from an individual, then the same fraction must also be taken from all non-poorer individuals. Symmetrically, if an individual's endowment is increased by a given proportion, then it must be the same for all non-richer individuals. As mentioned above, uniform proportional progressive transfers and the relative differentials quasi-ordering are connected as the following result makes clear.

Proposition 3.1. *Let $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}$. The following two statements are equivalent:*

(a) $\mathbf{w}^*$ *is obtained from* $\mathbf{w}^\circ$ *by means of a finite sequence of uniform proportional progressive transfers.*

(b) $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$ *and* $\nu(\mathbf{w}^*) = \nu(\mathbf{w}^\circ)$.

Not only does a uniform proportional progressive transfer reduces dispersion as measured by the relative differentials quasi-ordering, but also, if one distribution of productivities is less dispersed than another, then it can be obtained from the latter by means of a finite number of uniform proportional progressive transfers. We insist on the fact that for such an equivalence to hold, one has to make sure that the distributions of productivities have equal geometric means.[10] From a practical point of view, condition (b) may be seen as a convenient means

[10] Note however that, if $\nu(\mathbf{w}^*) \neq \nu(\mathbf{w}^\circ)$ and $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$, then it is possible to find a distribution $\tilde{\mathbf{w}}$ such that (i) $\mathbf{w}^* \geq_{RD} \tilde{\mathbf{w}}$ and $\nu(\mathbf{w}^*) = \nu(\tilde{\mathbf{w}})$ and (ii) $\tilde{\mathbf{w}} \sim_{RD} \mathbf{w}^\circ$. For this, it suffices to choose $\tilde{\mathbf{w}} = (\nu(\mathbf{w}^*)/\nu(\mathbf{w}^\circ))\,\mathbf{w}^\circ$.

for checking whether one distribution of the productivities can be obtained from another only by means of uniform proportional progressive transfers. Although it is open to debate, we think that the relative differentials quasi-ordering is more suited for comparing the degree of concentration of productivity levels than more standard inequality measures.[11]

For reasons that will become clear later, it is useful to consider those modifications of the allocation of productivities that unambiguously increase or decrease the productive efficiency of the economy. The following criterion, that allows us to compare distributions of productivities from the efficiency point of view, is quite natural.

Definition 3.3. Given two distributions of productivity $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}$, we say that $\mathbf{w}^*$ *is weakly more efficiently distributed than* $\mathbf{w}^\circ$, which we write $\mathbf{w}^* \geq_{ME} \mathbf{w}^\circ$, if and only if $w_i^* \geqslant w_i^\circ$, for all $i = 1, 2, \ldots, n$. Equivalently, we say that $\mathbf{w}^\circ$ *is less efficiently distributed than* $\mathbf{w}^*$, something we write $\mathbf{w}^\circ \geq_{LE} \mathbf{w}^*$.

The reader will rightly notice that our definition of a (weakly) more efficient distribution of productivities is actually nothing else than first order stochastic dominance or, equivalently, quantile dominance. On some occasion we will be willing to investigate the joint impact on the distribution of consumption of less dispersed and more or less efficiently allocated productivities. Our first additional criterion combines considerations for more efficiency and less dispersion.

Definition 3.4. Given two distributions of productivity $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}$, we say that $\mathbf{w}^*$ *is more efficiently distributed and less dispersed than* $\mathbf{w}^\circ$, which we write $\mathbf{w}^* \geq_{MERD} \mathbf{w}^\circ$, if and only if $\mathbf{w}^* \geq_{ME} \mathbf{w}^\circ$ and $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$.

For our second additional criterion, less dispersion goes along with less efficiently allocated productivities.

Definition 3.5. Given two distributions of productivity $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}$, we say that $\mathbf{w}^*$ *is less efficiently distributed and less dispersed than* $\mathbf{w}^\circ$, which we write $\mathbf{w}^* \geq_{LERD} \mathbf{w}^\circ$, if and only if $\mathbf{w}^* \geq_{LE} \mathbf{w}^\circ$ and $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$.

Whereas they are far from exhausting all the possible modifications of the distribution of productivities among the individuals, we believe that the above criteria allow to capture the essential features of the changes that may take place in practice.

4. Relative Inequality and Comparisons of Distributions of Income

Here, we are interested in the comparisons of the distributions of the agents' earnings at the market equilibria of two different artisan economies. We consider successively three cases:

[11] As is well-known, the variance of the logarithms violates the principle of transfers according to which a progressive transfer reduces inequality (see Foster and Ok (1999)). In this respect, it is worth noting that the variance of the logarithms is coherent with the relative differentials quasi-ordering provided that the distributions have the same geometric mean. In other words, dispersion as measured by the variance of the logarithms always decreases as the result of a uniform proportional progressive transfer (see Section A.5 for details).

(i) both economies have the same preferences but different distributions of individual productivities, (ii) the economies have different preferences but the distributions of individual productivities are identical, and (iii) the economies differ both in terms of their preferences and distributions of productivities. We first have to make precise how we compare the distributions of income from an inequality point of view.

4.1. The Measurement of Relative Inequality

As we indicated in the Introduction, we want to obtain results that do not depend on a particular inequality index but hold rather for a large spectrum of value judgements. To this end, we appeal to the *relative Lorenz criterion* in order to make comparisons of inequality within the same society or across different societies. The ordinate of the *relative Lorenz curve* at $p = k/n$ of the consumption distribution $\mathbf{c} := (c_1, \ldots, c_n)$ such that $0 < c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$ is given by[12]

$$RL\left(\frac{k}{n};\mathbf{c}\right) := \frac{1}{n}\sum_{j=1}^{k}\frac{c_j}{\mu(\mathbf{c})},\ \forall k = 1,2,\ldots,n, \tag{4.1}$$

where $\mu(\mathbf{c})$ is the *arithmetic mean* of distribution $\mathbf{c} := (c_1, \ldots, c_n)$. A consumption distribution is then considered as less unequal than another if its relative Lorenz curve lies nowhere below that of the latter, which we formally state as follows:

Definition 4.1. Given two consumption distributions $\mathbf{c}^*, \mathbf{c}^\circ \in \mathbb{R}^n_{++}$, we say that $\mathbf{c}^*$ *relative Lorenz dominates* $\mathbf{c}^\circ$, which we write $\mathbf{c}^* \geq_{RL} \mathbf{c}^\circ$, if and only if:

$$RL\left(\frac{k}{n};\mathbf{c}^*\right) \geqslant RL\left(\frac{k}{n};\mathbf{c}^\circ\right),\ \forall k = 1,2,\ldots,(n-1). \tag{4.2}$$

Recourse to the relative Lorenz criterion ensures that one distribution cannot be considered more equal than another if it is ranked below the latter by at least one reasonable (relative) inequality index. In other words, all inequality indices that are deemed relevant must agree on the ranking of the pair of distributions under comparison for one distribution to dominate another according to the relative Lorenz quasi-ordering. The class of inequality indices consistent with the relative Lorenz quasi-ordering is quite large and contains most of the inequality measures currently used such as the Gini index, the Atkinson-Kolm-Sen (AKS) family of indices as well as the generalised entropy family. All these indices have the property that a progressive transfer – the operation consisting of transferring part of the consumption of a rich individual to a poorer one – reduces inequality. Another important property, that will play a crucial role in the proofs of some results, is the fact that equiproportionate additions leave inequality unchanged. For more details, the reader is referred to the surveys by Foster (1985) or Moyes (1999) and the references therein.

[12] There is no loss of generality to restrict attention to consumption vectors that are non-decreasingly arranged given the Spence-Mirrlees condition and the assumption that individuals are labelled according to their productivities.

4.2. Less Dispersed Productivities Do Not Reduce Inequality

What is the impact on income inequality of a change in the dispersion of productivities when all the agents in the same society have identical preferences? It is perhaps not totally unreasonable to expect that, in such a situation, less dispersed productivities might lead to a reduction of income inequality. However, income inequality does not necessarily decrease as a result of more concentrated productivities: there are indeed situations where *some admissible indices* record an increase in consumption inequality. However, it is possible to find situations where a reduction of the dispersion of productivities does not move the relative Lorenz curve of the distribution of income upwards, in which case *some admissible indices* will record an increase in inequality. Or, it may even be the case that a reduction in the dispersion of individual productivities generates an *unambiguous* increase in consumption inequality.

Maybe the simplest way to convince the reader that preferences play a role is to examine what happens in particular – though not totally unrealistic – situations. In the following example, we are interested in the impact on consumption inequality of particular transformations of the distributions of productivities that *all* have the property that dispersion is reduced.

Example 4.1. Consider the three following utility functions that all belong to the class $\mathscr{U}$:[13]

$$u^{(1)}(c,\ell) := c - \frac{\ell^2}{2};$$

$$u^{(3)}(c,\ell) := \ln c - \frac{1}{c} - \ell; \text{ and}$$

$$u^{(9)}(c,\ell) := \int_0^c \frac{1}{H^{-1}(s)} ds - \ell, \text{ where } H(w) = \frac{w^2}{(w^2 + (4 - w^2))^{\frac{1}{2}}}.$$

Table 4.1 indicates how consumption inequality reacts to reductions of the dispersion of productivities in particular situations. More precisely, we distinguish three types of transformations of the distribution of productivities: $\mathbf{w}^*$ is obtained from $\mathbf{w}^\circ$ by means of a single uniform proportional progressive transfer (Case 1); $\mathbf{w}^*$ is less dispersed and more efficient than $\mathbf{w}^\circ$ (Case 2); and $\mathbf{w}^*$ is less dispersed and less efficient than $\mathbf{w}^\circ$ (Case 3).

Table 4.1: Impact of less dispersed productivities on consumption inequality

		$C(\mathbf{w}^*;u) >_{RL} C(\mathbf{w}^\circ;u)$	$C(\mathbf{w}^*;u) <_{RL} C(\mathbf{w}^\circ;u)$	$\neg[C(\mathbf{w}^*;u) \gtrless_{RL} C(\mathbf{w}^\circ;u)]$ [14]
Case 1	$\mathbf{w}^\circ = (0.2, 0.5, 3.0, 5.0)$ $\mathbf{w}^* = (0.4, 1.0, 1.5, 2.5)$	$u^{(1)}(c,\ell)$	na	$u^{(9)}(c,\ell)$
Case 2	$\mathbf{w}^\circ = (2.00, 6.00)$ $\mathbf{w}^* = (1.50, 4.00)$	$u^{(3)}(c,\ell)$	$u^{(9)}(c,\ell)$	na
Case 3	$\mathbf{w}^\circ = (0.20, 0.40)$ $\mathbf{w}^* = (0.63, 1.17)$	$u^{(9)}(c,\ell)$	$u^{(3)}(c,\ell)$	na

[13] We have chosen separable utility functions for making computations easier but similar results can be obtained with non-separable utility functions.

[14] The notation $C(\mathbf{w}^*;u) \gtrless_{RL} C(\mathbf{w}^\circ;u)$ is intended to mean that, *either* $C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)$, *or* $C(\mathbf{w}^*;u) \leq_{RL} C(\mathbf{w}^\circ;u)$.

Case 1 is an instance where a uniform proportional progressive transfer, *either* results in an unambiguous decrease of consumption inequality if the agents' utility function is $u^{(1)}$, *or* gives rise to a distribution of consumption that cannot be compared with the initial distribution of consumption by the relative Lorenz quasi-ordering if the agents make decisions on the basis of the utility function $u^{(9)}$. In the latter event, this means that it is always possible to find two inequality indices such that one index ranks distribution $C(\mathbf{w}^*, u^{(9)})$ above distribution $C(\mathbf{w}^\circ, u^{(9)})$, while the other index provides the opposite ranking. Cases 2 and 3 are concerned with the impact of a reduction of the dispersion of productivities coupled with less efficiently and more efficiently distributed productivities, respectively. Again, it is possible to obtain an increase in inequality or a decrease in inequality by choosing appropriately the utility functions.

The preceding example demonstrates that preferences are important: depending of which preferences are selected, a given transformation of the distribution of productivities that reduces their dispersion may result in an unambiguous decrease or increase of consumption inequality. Or, eventually, the impact on the distribution of consumption is unclear in the sense that it is always possible to find an index according to which inequality has increased. The example also shows that, not only is the choice of the preference ordering important, but also the shape of the selected preferences. Indeed, different transformations of the distribution of productivities, that all reduce dispersion, may have different impacts on consumption inequality even under given preferences. This is illustrated by the utility function $u^{(9)}$, for then, depending on which transformations are considered, consumption inequality may decrease (Case 3), or increase (Case 2), or be ambiguous (Case 1).

To sum up, this example suggests that, for income inequality to decrease as the result of less dispersed productivities, further restrictions have to be imposed on the agents' common preference ordering. However, the direct identification of these additional properties of the preference ordering proves to be a difficult task and we rather follow here a different route. More precisely, we search for the properties of the consumption function that guarantee that a reduction in the dispersion of the productivities always imply that incomes become more equally distributed. There is no loss of generality in proceeding in this way since, in our particular model, the consumption function uniquely represents the preference ordering. We come back to this issue in Section 6 where we identify the properties of the utility functions that guarantee that income inequality decreases in the special case where preferences are quasi-linear.

4.3. Identical Preferences and Different Distributions of Productivities

The elasticity of the consumption function with respect to productivity will prove to be the key factor for signing the impact on income inequality of changes in the distribution of productivities. To simplify notation and for later use, we denote respectively by $C'(w;u)$ and $\eta(C,w;u) := C'(w;u)\,w/C(w;u)$ the first derivative and the elasticity of the consumption function $C(\cdot\,;u)$ evaluated at productivity w. Our first result identifies those consumption functions with the property that consumption is more equally distributed among the agents as productivity becomes more concentrated.

Proposition 4.1. *Let $u \in \mathscr{U}$. The following two statements are equivalent:*

(a) *For all $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}(u)$; $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$ implies $C(\mathbf{w}^*; u) \geq_{RL} C(\mathbf{w}^\circ; u)$.*

(b) *$\eta(C, w; u)$ is constant in w, for all $w \in W(u)$.*

According to Proposition 4.1, a constant consumption elasticity is both necessary and sufficient for consumption inequality to be unambiguously reduced as the result of less dispersed productivities. Requiring that consumption inequality decreases when productivities become less dispersed without imposing additional constraints is a very strong condition and it is therefore not surprising that one is left with a particularly small class of consumption functions.

To which extent is the introduction of additional restrictions on the distributions of productivities likely to affect the properties of the consumption function that ensure inequality reduction? For instance, do we get a larger class of consumption functions if we impose in addition that the reduction of dispersion preserves the geometric mean of the distribution of productivities, which, invoking Proposition 3.1, is equivalent to the fact that one distribution of productivities is obtained from another by means of a sequence of uniform proportional progressive transfers? The following result confirms that a constant elasticity of the consumption function is a sufficient condition for a uniform proportional progressive transfer to result in a more equal distribution of consumption.

Proposition 4.2. *Let $u \in \mathscr{U}$. Then, statement* (b) *implies statement* (a).

(a) *For all $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}(u)$; $C(\mathbf{w}^*; u) \geq_{RL} C(\mathbf{w}^\circ; u)$ whenever $\mathbf{w}^*$ is obtained from $\mathbf{w}^\circ$ by means of uniform proportional progressive transfers.*

(b) *$\eta(C, w; u)$ is constant in w, for all $w \in W(u)$.*

However, Proposition 4.2 provides no answer to the question to know whether the constancy of consumption elasticity is also necessary for the agents' consumptions to be more equally distributed. Actually, it can be shown – under additional but weak restrictions – that the constant elasticity condition is both necessary and sufficient for less dispersed consumptions to obtain as the result of uniform proportional progressive transfers.[15]

The constancy of the consumption elasticity with respect to productivity, which is particularly restrictive, is actually equivalent to requiring that

$$\frac{C(\lambda w^*; u)}{C(w^*; u)} = \frac{C(\lambda w^\circ; u)}{C(w^\circ; u)}, \; \forall\, \lambda > 1, \; \forall\, w^*, w^\circ, \lambda w^*, \lambda w^\circ \in W(u), \tag{4.3}$$

a functional equation whose solution (Aczel, 1966, Chapter 3) is

$$C(w; u) = \gamma w^\beta \; (\gamma, \beta > 0), \; \forall\, w \in W(u). \tag{4.4}$$

For sure, the meaning of the requirement of a constant consumption elasticity might look rather abstruse at first sight. In this respect, the equivalent condition (4.3) makes more explicit how demanding condition (b) of Proposition 4.1 is. According to this condition, a proportional

[15] We provide in Section A.4 a proof of this assertion that builds on arguments different from those used for proving Proposition 4.2.

change in an agent's productivity must imply a proportional change in her consumption, though not necessarily of the same extent.[16]

So far we have not exploited the information according to which productivities might be more efficiently distributed in one society than in another. Here we are interested in the impact on income inequality of a change in the dispersion of productivities that is accompanied by an increase or a decrease of productive efficiency. Substituting an increase or a decrease in efficiency for the equality of the geometric mean results in a substantial weakening of the restrictions that have to be placed on the consumption function in order that inequality decreases. The following result provides the answer for the case where less dispersion goes hand in hand with more efficiently distributed productivities.

Proposition 4.3. *Let $u \in \mathscr{U}$. The following two statements are equivalent:*

(a) *For all $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}(u)$; $\mathbf{w}^* \geq_{MERD} \mathbf{w}^\circ$ implies $C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)$.*

(b) *$\eta(C,w;u)$ is non-increasing in w, for all $w \in W(u)$.*

Proposition 4.3 shows how the class of consumption functions expands when one imposes the additional requirement that productivities are more efficiently distributed. Condition (b) of Proposition 4.3 is reminiscent of the concept of *increasing average progression* encountered in the taxation literature (see, e.g., Jakobsson (1976), Le Breton *et al.* (1996), Lambert (2001)). It can be equivalently stated as

$$\text{(4.5)} \qquad \frac{C(\lambda w^*;u)}{C(w^*;u)} \leqslant \frac{C(\lambda w^\circ;u)}{C(w^\circ;u)},\ \forall\, \lambda > 1,\ \forall\, w^* > w^\circ \text{ such that } w^*, w^\circ, \lambda w^*, \lambda w^\circ \in W(u),$$

according to which the relative increase in an agent's consumption caused by a proportional increase in her productivity diminishes with productivity. If now we want income inequality to decrease when productivities are less dispersed but also less efficiently distributed, then we obtain the following result that does not come as a surprise.

Proposition 4.4. *Let $u \in \mathscr{U}$. The following two statements are equivalent:*

(a) *For all $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}(u)$; $\mathbf{w}^* \geq_{LERD} \mathbf{w}^\circ$ implies $C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)$.*

(b) *$\eta(C,w;u)$ is non-decreasing in w, for all $w \in W(u)$.*

The non-decreasingness of the consumption function elasticity guarantees that income inequality will always decrease as the agents' productivities become less dispersed and, at the same time, less efficiently distributed.

It is tempting to relate the conditions identified in Proposition 4.3 (resp. Proposition 4.4) to the concavity (resp. convexity) of the consumption function. Actually, the concavity of the consumption function and the requirement that its elasticity be non-increasing are

[16] The only case where consumption increases or decreases in the same proportion as productivity is when $\eta(C,w;u) = 1$. This implies that consumption is proportional to productivity, which happens for instance in the case of the utility function $u^{(4)}(c,\ell) = \ln c - \ell$.

independent properties. Making use of the derivative of the consumption elasticity, condition (b) of Propositions 4.3 can be equivalently rewritten as:

$$\eta(C', w; u) \leqslant \eta(C, w; u) - 1, \; \forall\, w \in W(u), \tag{4.6}$$

where $\eta(C', w; u)$ is the elasticity of the first derivative of the consumption function $C(\cdot\,; u)$. In the absence of additional restrictions, there is no logical relationship between the concavity of the consumption function and the fact that it has a non-increasing elasticity.[17] Similarly, the convexity of the consumption function does not guarantee that its elasticity is non-decreasing. An example is provided by the utility function $u^{(2)}(c, \ell) := c - e^{\ell}$. Then, we have $C(w; u^{(2)}) = w \ln w$, which is convex, but $\eta(C, w; u^{(2)}) = (1 + \ln w)/\ln w$ is decreasing over the interval $W(u^{(2)}) := (1, +\infty)$. Under convex consumption, less dispersed and less efficiently allocated productivities would not necessarily result in a decrease in income inequality. Thus, the convexity or concavity of the consumption function is no impediment for its elasticity to increase or decrease, and conversely.

Propositions 4.1, 4.3 and 4.4 make clear that the efficiency dimension of the transformation of the distribution of productivities plays a crucial role in the determination of the classes of the consumption functions with the property that income inequality decreases when productivities are less dispersed. Conditions (b) of Propositions 4.1, 4.3 and 4.4 implicitly constrain the underlying preference orderings and this raises immediately two questions. Do there exist preference orderings that generate consumption functions that verify these conditions? If so, what do these preference orderings look like? A look at Table 4.2 should convince the reader that the conditions the consumption function needs to satisfy in order that inequality decreases when productivities are more concentrated are not totally unrealistic. There exist non-pathological utility functions – and therefore preferences – that generate such consumption patterns. It is also worth noting that these elasticity conditions are compatible with a large

Table 4.2: Propositions 4.1, 4.3 and 4.4 in a glance

	$C(\mathbf{w}^*; u) \geq_{RL} C(\mathbf{w}^\circ; u)$	
$\mathbf{w}^* \geq_J \mathbf{w}^\circ$	Consumption Function	Utility Function
$\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$	$\eta_w(C, w; u) = 0$	$u^{(1)}(c, \ell) = c - \dfrac{\ell^2}{2}$ $u^{(11)}(c, \ell) = -(\ell + 3)e^{-\frac{c-3}{\ell+3}} - 1$
$\mathbf{w}^* \geq_{MERD} \mathbf{w}^\circ$	$\eta_w(C, w; u) < 0$	$u^{(2)}(c, \ell) = c - e^{\ell}$ $u^{(12)}(c, \ell) = -(\ell + 3)e^{-\frac{c-2}{\ell+3}} - 1$
$\mathbf{w}^* \geq_{LERD} \mathbf{w}^\circ$	$\eta_w(C, w; u) > 0$	$u^{(8)}(c, \ell) = c - \dfrac{5}{2}\left[\ell e^{\frac{-1}{\ell}} - \displaystyle\int_1^{+\infty} \frac{e^{-t\ell}}{t}\, dt\right]$ $u^{(13)}(c, \ell) = -(\ell + 3)e^{-\frac{c-4}{\ell+3}} - 1$

[17] For instance, if one imposes the further restriction that $0 \leqslant \eta(C, w; u) < 1$, then condition (4.6), in conjunction with the Spence-Mirrlees condition, implies that the consumption function is concave. On the other hand, $u^{(3)}(c, \ell) = \ln c - (1/c) - \ell$ is an example of a utility function that generates a consumption function that is concave but whose elasticity is increasing everywhere.

class of preference orderings that comprise, among other things, quasi-linear preferences (e.g., utility functions $u^{(1)}$, $u^{(2)}$ and $u^{(8)}$) and non-separable preferences (e.g., utility functions $u^{(11)}$, $u^{(12)}$ and $u^{(13)}$). The question of whether these conditions are plausible or likely to be met in practice is something that remains to be investigated but that lies outside the scope of this paper.[18]

4.4. Different Preferences and Identical Distributions of Productivities

We consider now the impact on income inequality of a modification of the preference ordering which, as before, we assimilate with a modification of the consumption function. We have seen in Example 4.1 that a given transformation of the distribution of productivities that reduces dispersion can have different impacts on the distribution of consumption depending on which utility function is used. Before addressing the general question of the impact of simultaneous modifications of the preferences and of the distribution of productivities (see Section 4.5 below), we would like to examine how preferences affect consumption inequality when the distribution of productivities is kept fixed.

The following result identifies the condition that has to verified by two consumption functions in order that income inequality decreases, whatever the way productivities are allocated among the agents, when one consumption function is substituted for the other.

Proposition 4.5. *Let $\tilde{u}$ and $\hat{u} \in \mathscr{U}$. The following two statements are equivalent:*

(a) *For all $\mathbf{w} \in \mathscr{W}(\tilde{u}) \cap \mathscr{W}(\hat{u})$; $C(\mathbf{w};\tilde{u}) \geq_{RL} C(\mathbf{w};\hat{u})$.*

(b) *$\eta(C,w;\tilde{u}) \leqslant \eta(C,w;\hat{u})$, for all $w \in W(\tilde{u}) \cap W(\hat{u})$.*

According to Proposition 4.5 a necessary and sufficient condition for income inequality to decrease as the result of a shift in the preferences is that the elasticity of the new consumption function is nowhere greater than the elasticity of the original consumption function. Condition (b) in Proposition 4.5 is reminiscent of the concept of a *more progressive tax schedule* encountered in the taxation literature (see, e.g., Jakobsson (1976), Le Breton *et al.* (1996) or Lambert (2001) among others). It can be stated equivalently as

$$\frac{C(\lambda w;\tilde{u})}{C(w;\tilde{u})} \leqslant \frac{C(\lambda w;\hat{u})}{C(w;\hat{u})},\ \forall\, \lambda > 1,\ \forall\, w, \lambda w \in W(\tilde{u}) \cap W(\hat{u}), \tag{4.7}$$

which means that the relative increase in consumption caused by a proportional increase in productivity is smaller under $C(\,\cdot\,;\tilde{u})$ than under $C(\,\cdot\,;\hat{u})$. Here again it is possible to provide pairs of utility functions such that the corresponding consumption functions satisfy condition (b) of Proposition 4.5: two such pairs are presented in Table 4.3. We note that this elasticity condition does not impose stringent constraints on the underlying preference orderings: in particular, quasi-linear as well as non-separable preferences are equally admissible.

[18] Leaving aside the fact that productivities are difficult to observe, the major problem is that the consumption patterns one observes in practice are determined jointly by the distribution of productivities and by different institutional arrangements. This makes it difficult to separate the changes in the distribution of consumption that stem from modifications in the allocation of productivities from those that result, for instance, from modifications of the tax system, all other things being the same.

Table 4.3: Proposition 4.5 in a glance

$C(\mathbf{w};\tilde{u}) \geq_{RL} C(\mathbf{w};\hat{u})$		
Consumption Functions	Utility Function $\tilde{u}(c,\ell)$	Utility Function $\hat{u}(c,\ell)$
$\eta(C,w;\tilde{u}) \leqslant \eta(C,w;\hat{u})$	$u^{(10)}(c,\ell) = c - \frac{c^2}{8} - \ell$ $u^{(13)}(c,\ell) = -(\ell+3)e^{-\frac{c-4}{\ell+3}-1}$	$u^{(6)}(c,\ell) = -e^{-c} - \ell$ $u^{(12)}(c,\ell) = -(\ell+3)e^{-\frac{c-2}{\ell+3}-1}$

4.5. Different Preferences and Different Distributions of Productivities

Finally, we turn to the general case where both the preferences and the distributions of productivities are allowed to vary simultaneously. Suppose that productivities become less dispersed: what changes in preferences guarantee that income inequality is reduced as a result? Or conversely, which transformations of the distribution of productivities ensure that the inequality reducing impact of the change in the preferences is preserved? As Example 4.1 showed, the impact on consumption inequality of more concentrated productivities is highly conditioned by the agents' common preferences. Depending on the chosen preference ordering, a given reduction o the dispersion of productivities can give rise to an unambiguous decrease or increase of inequality, or eventually to a distribution of consumption that cannot be compared with the initial one. This suggests that, in the absence of appropriate restrictions on the way the distribution of productivities and the preferences are simultaneously altered, almost anything can occur as far as income inequality is concerned.

Since both the preferences and the distributions of productivities change at the same time, a natural way to go is to examine what we can learn from our previous results. It is a direct consequence of Propositions 4.3 and 4.5 that, for $C(\mathbf{w}^*;\tilde{u}) \geq_{RL} C(\mathbf{w}^\circ;\hat{u})$ whenever $\mathbf{w}^* \geq_{MERD} \mathbf{w}^\circ$, it is sufficient that

$$\eta(C,w;\tilde{u}) \leqslant \eta(C,w;\hat{u}),\ \forall\, w \in W(\tilde{u}) \cap W(\hat{u}),\ \text{and} \tag{4.8}$$

$$\eta(C,w;\tilde{u}) \text{ is non-increasing in } w,\ \forall\, w \in W(\tilde{u}) \cap W(\hat{u}). \tag{4.9}$$

Indeed, assuming that $\mathbf{w}^* \geq_{MERD} \mathbf{w}^\circ$ and invoking Proposition 4.3 and condition (4.8), we conclude that $C(\mathbf{w}^*;\tilde{u}) \geq_{RL} C(\mathbf{w}^\circ;\tilde{u})$. Then, condition (4.9) and Proposition 4.5 together ensure that $C(\mathbf{w}^\circ;\tilde{u}) \geq_{RL} C(\mathbf{w}^\circ;\hat{u})$, and the transitivity of the relative Lorenz quasi-ordering makes the argument complete. Similarly, Propositions 4.4 and 4.5 guarantee that, if

$$\eta(C,w;\hat{u}) \text{ is non-decreasing in } w,\ \forall\, w \in W(\tilde{u}) \cap W(\hat{u}), \tag{4.10}$$

and condition (4.9) holds, then $C(\mathbf{w}^*;\tilde{u}) \geq_{RL} C(\mathbf{w}^\circ;\hat{u})$ whenever $\mathbf{w}^* \geq_{LERD} \mathbf{w}^\circ$.

However, while these conditions are sufficient for income inequality to decrease, they are far from being necessary as the following example demonstrates.

Example 4.2. Consider the utility functions $u^{(2)}(c,\ell) = c - e^{\ell}$ and $u^{(3)}(c,\ell) = \ln c - (1/c) - \ell$. While we have

$$\eta(C,w;u^{(3)}) < \eta(C,w;u^{(2)}),\ \text{for all } w \in W(^{(3)}) \cap W(^{(2)}) = (1,+\infty), \tag{4.11}$$

it must be noted that $\eta(C, w; u^{(3)})$ is increasing over $(1, +\infty)$. Choose the distributions of productivities $\mathbf{w}^\circ = (1.50, 3.50)$ and $\mathbf{w}^* = (2.00, 4.00)$ so that $\mathbf{w}^* >_{MERD} \mathbf{w}^\circ$. At the market equilibrium, we obtain $C(\mathbf{w}^*; u^{(3)}) \approx (2.73, 4.83)$ and $C(\mathbf{w}^\circ; u^{(2)}) \approx (0.61, 4.38)$, and it can be checked that $C(\mathbf{w}^*; u^{(3)}) >_{RL} C(\mathbf{w}^\circ; u^{(2)})$. Suppose now that distribution $\mathbf{w}^\circ$ is transformed into distribution $\mathbf{w}^* = (1.00, 2.00)$ so that $\mathbf{w}^* >_{LERD} \mathbf{w}^\circ$. Then we obtain $C(\mathbf{w}^*; u^{(3)}) \approx (1.62, 2.73)$, and it follows that $C(\mathbf{w}^*; u^{(3)}) >_{RL} C(\mathbf{w}^\circ; u^{(2)})$, even though $\eta(C, w; u^{(2)})$ is decreasing over $(1, +\infty)$.

The above example should convince the reader that conditions (4.8) and (4.9) – similarly conditions (4.8) and (4.10) – taken together may be too strong and that it should be possible to find weaker restrictions that still guarantee that income inequality decreases when productivities become more concentrated and preferences change at the same time. The next result provides the answer in the absence of any information about the direction of the change in efficiency implied by the increase in the concentration of productivities.

Proposition 4.6. *Let $\tilde{u}$ and $\hat{u} \in \mathscr{U}$. The following two statements are equivalent:*

(a) *For all $\mathbf{w} \in \mathscr{W}(\tilde{u}) \cap \mathscr{W}(\hat{u})$; $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$ implies $C(\mathbf{w}^*; \tilde{u}) \geq_{RL} C(\mathbf{w}^\circ; \hat{u})$.*

(b) *$\eta(C, w; \tilde{u}) \leqslant \eta(H, w) \leqslant \eta(C, w; \hat{u})$, for all $w \in W(\tilde{u}) \cap W(\hat{u})$ and some differentiable function H with constant elasticity.*

Proposition 4.6 confirms that for income inequality to decrease as the result of less dispersed productivities, it is necessary that the elasticity of $C(\cdot; \tilde{u})$ is not greater than the elasticity of $C(\cdot; \hat{u})$ over the relevant domain. But it also underlines the fact that this is not sufficient: upper and lower bounds are respectively imposed on the elasticities of $C(\cdot; \tilde{u})$ and $C(\cdot; \hat{u})$, and these bounds are constant over the admissible range of productivities. Condition (b) of Proposition 4.6 actually requires that the curves representing the elasticities of the consumption functions can be separated by an horizontal line. On the other hand, Proposition 4.6 also confirms that it is not necessary that $C(\cdot; \tilde{u})$ or $C(\cdot; \hat{u})$ verifies conditions (4.9) or (4.10) for income inequality to decrease.

So far we have searched for those adjustments of the preferences that ensure that a reduction of the dispersion of productivities give rise to a decrease in income inequality without exploiting information about changes in productive efficiency. To which extent is this additional information likely to change the necessary and sufficient conditions identified in Proposition 4.6? The next result gives the answer when one is interested in the impact on income inequality of a change in the dispersion of productivities accompanied by an increase in efficiency.

Proposition 4.7. *Let $\tilde{u}$ and $\hat{u} \in \mathscr{U}$. The following two statements are equivalent:*

(a) *For all $\mathbf{w} \in \mathscr{W}(\tilde{u}) \cap \mathscr{W}(\hat{u})$; $\mathbf{w}^* \geq_{MERD} \mathbf{w}^\circ$ implies $C(\mathbf{w}^*; \tilde{u}) \geq_{RL} C(\mathbf{w}^\circ; \hat{u})$.*

(b) *$\eta(C, w; \tilde{u}) \leqslant \eta(H, w) \leqslant \eta(C, w; \hat{u})$, for all $w \in W(\tilde{u}) \cap W(\hat{u})$ and some differentiable function H with non-increasing elasticity.*

The fact that productivities are more efficiently distributed allows us to weaken the conditions to be fulfilled by the consumption functions $C(\cdot; \tilde{u})$ and $C(\cdot; \hat{u})$. Still there are upper and lower bounds imposed on the elasticities of the consumption functions $C(\cdot;\tilde{u})$ and $C(\cdot;\hat{u})$, respectively, but now these bounds are no longer constant: they decline with the level of productivity. Symmetrically, if it happens that productivities are less efficiently distributed while they are at the same time more concentrated, then we have the following result that does not come as a surprise.

Proposition 4.8. *Let $\tilde{u}$ and $\hat{u} \in \mathscr{U}$. The following two statements are equivalent:*

(a) *For all $\mathbf{w} \in \mathscr{W}(\tilde{u}) \cap \mathscr{W}(\hat{u})$; $\mathbf{w}^* \geq_{LERD} \mathbf{w}^\circ$ implies $C(\mathbf{w}^*;\tilde{u}) \geq_{RL} C(\mathbf{w}^\circ;\hat{u})$.*

(b) *$\eta(C,w;\tilde{u}) \leqslant \eta(H,w) \leqslant \eta(C,w;\hat{u})$, for all $w \in W(\tilde{u}) \cap W(\hat{u})$ and some differentiable function H with non-decreasing elasticity.*

If it is possible to find a function whose elasticity is everywhere non-decreasing, nowhere below the elasticity of $C(\cdot;\tilde{u})$ and nowhere above the elasticity of $C(\cdot;\hat{u})$, then more concentrated

Table 4.4: Propositions 4.6, 4.7 and 4.8 in a glance

$\mathbf{w}^* \geq_J \mathbf{w}^\circ$	$C(\mathbf{w}^*;\tilde{u}) \geq_{RL} C(\mathbf{w}^\circ;\hat{u})$		
	Consumption Function	$\tilde{u}(c,\ell)$	$\hat{u}(c,\ell)$
$\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$	$\eta(C,w;\tilde{u}) \leqslant \eta(H,w) \leqslant \eta(C,w;\hat{u})$ $\eta_w(H,w) = 0$	$u^{(4)}(c,\ell) = \ln c - \ell$ $u^{(3)}(c,\ell) = \ln c - \dfrac{1}{c} - \ell$	$u^{(5)}(c,\ell) = 2\sqrt{c} - \ell$ $u^{(2)}(c,\ell) = c - e^\ell$
$\mathbf{w}^* \geq_{MERD} \mathbf{w}^\circ$	$\eta(C,w;\tilde{u}) \leqslant \eta(H,w) \leqslant \eta(C,w;\hat{u})$ $\eta_w(H,w) \leqslant 0$	$u^{(4)}(c,\ell) = \ln c - \ell$ $u^{(10)}(c,\ell) = c - \dfrac{c^2}{8} - \ell$	$u^{(5)}(c,\ell) = 2\sqrt{c} - \ell$ $u^{(6)}(c,\ell) = -e^{-c} - \ell$
$\mathbf{w}^* \geq_{LERD} \mathbf{w}^\circ$	$\eta(C,w;\tilde{u}) \leqslant \eta(H,w) \leqslant \eta(C,w;\hat{u})$ $\eta_w(H,w) \geqslant 0$	$u^{(4)}(c,\ell) = \ln c - \ell$ $u^{(3)}(c,\ell) = \ln c - \dfrac{1}{c} - \ell$	$u^{(5)}(c,\ell) = 2\sqrt{c} - \ell$ $u^{(2)}(c,\ell) = c - e^\ell$

and less efficient productivities always result in an unambiguous decrease in income inequality, and conversely.

The necessary and sufficient conditions identified in Propositions 4.6, 4.7 and 4.8 impose constraints on the preferences orderings of the two economies under comparison. While it is a difficult exercise to uncover the precise meaning of these restrictions for the shape of the preference orderings (see however Section 6 below), it is easy to convince oneself of the existence of such preferences by inspection of Table 4.4 where we provide instances of pairs of utility functions that fulfill these conditions.

5. Absolute Inequality and Comparisons of Distributions of Income

The inequality criterion The concept of relative inequality has been challenged by certain authors (see in particular Kolm (1976)) and this has given rise to different alternatives to the relative Lorenz quasi-ordering in the literature (see, among others, Bossert and Pfingsten (1990), Krtscha (1994), Del Rio and Ruiz-Castillo (2000)). Here, we follow Kolm (1976)'s original suggestion and take the view that it is the absolute rather than the relative differences in consumption levels that matter. There are admittedly other possibilities and it is in principle possible to adapt the results in this section to conform with these alternative views. The analogue to the relative Lorenz quasi-ordering when one subscribes to Kolm (1976)'s proposal is the so-called *absolute Lorenz criterion.* While the standard relative Lorenz criterion compares the cumulated consumption shares of the agents, the absolute Lorenz criterion is concerned with the cumulated consumption shortfalls from mean consumption of the agents in the economy (see Moyes (1987)). More precisely, the ordinate of the *absolute Lorenz curve* at $p = k/n$ of the consumption distribution $\mathbf{c} := (c_1, \dots, c_n)$ such that $0 < c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$ is given by

$$AL\left(\frac{k}{n}; \mathbf{c}\right) := \frac{1}{n} \sum_{j=1}^{k} \left[c_j - \mu(\mathbf{c})\right], \; \forall k = 1, 2, \dots, n. \tag{5.1}$$

It follows that $AL\,(k/n; \mathbf{c}) \leqslant 0$, for all $k = 1, 2, \dots, n-1$, and $AL\,(1; \mathbf{c}) = 0$, with strict inequalities whenever there exists at least one $i \in \{1, 2, \dots, n\}$ such that $c_i \neq \mu(\mathbf{c})$. The quantity $-n\,AL\,(k/n; \mathbf{c})$ represents the amount of consumption needed in order to ensure to each of the k poorest individuals a consumption level equal to mean consumption. The absolute Lorenz quasi-ordering is based on the comparison of the absolute Lorenz curves and it is formally defined as follows.

Definition 5.1. Given two consumption distributions $\mathbf{c}^*, \mathbf{c}^\circ \in \mathbb{R}^n_{++}$, we say that $\mathbf{c}^*$ *absolute Lorenz dominates* $\mathbf{c}^\circ$, which we write $\mathbf{c}^* \geq_{AL} \mathbf{c}^\circ$, if and only if:

$$AL\left(\frac{k}{n}; \mathbf{c}^*\right) \geqslant AL\left(\frac{k}{n}; \mathbf{c}^\circ\right), \; \forall k = 1, 2, \dots, (n-1). \tag{5.2}$$

Less dispersed productivities and income inequality As in Section 4, we would like to know in what circumstances less dispersed productivities give rise to less unequally distributed consumption levels when the preference ordering is given. More generally, we are interested in those adjustments of the preferences that guarantee that a reduction in the dispersion of productivities translates into a reduction in income inequality as measured by the absolute Lorenz criterion. Since the conditions we obtain are to a large extent straightforward adaptations of the conditions derived in Section 4, we avoid presenting a long list of propositions and we rather summarise the main results by means of tables.

It has been shown in Section 4 that the elasticity of the consumption function is the key factor for appraising the impact on relative inequality of a reduction in the dispersion of productivities. The derivative of the consumption function with respect to the logarithm of productivity plays a similar role when the focus is on absolute inequality. To simplify notation, we denote by $\xi(C, w; u) := C'(w; u)\, w$ the derivative of the consumption function with respect to the logarithm of productivity. Table 5.1 indicates the restrictions that must be placed on the consumption function in order that absolute consumption inequality declines as the result of less dispersed productivities when the preference ordering is fixed. When the consumption function is linear in the logarithm of productivity, income inequality is guaranteed to decrease as productivities become less dispersed. This is equivalent to requiring that the (absolute) changes in consumption caused by a proportional increase in productivity are independent

Table 5.1: Identical preferences and different distributions of productivities

$\mathbf{w}^* \geq_J \mathbf{w}^\circ$	$C(\mathbf{w}^*;\tilde{u}) \geq_{AL} C(\mathbf{w}^\circ;\hat{u})$	
	Consumption Function	Utility Function
$\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$	$\xi_w(C,w;u) = 0$	$u^{(6)}(c,\ell) = -e^c - \ell$
$\mathbf{w}^* \geq_{MERD} \mathbf{w}^\circ$	$\xi_w(C,w;u) < 0$	$u^{(10)}(c,\ell) = c - \dfrac{c^2}{8} - \ell \; (0 < c < 4)$
$\mathbf{w}^* \geq_{LERD} \mathbf{w}^\circ$	$\xi_w(C,w;u) > 0$	$u^{(7)}(c,\ell) = -e^{-c} - e^{\ell}$

from the agents' productivities. In other words, we must have

$$C(\lambda w^*;u) - C(w^*;u) = C(\lambda w^\circ;u) - C(w^\circ;u), \; \forall\, \lambda > 1, \; \forall\, w^*, w^\circ, \lambda w^*, \lambda w^\circ \in W(u), \tag{5.3}$$

a functional equation whose solution (Aczel (1966, Chapter 3)) is

$$C(w;u) = \alpha + \beta \ln w \; (\alpha \in \mathbb{R}, \beta > 0), \; \forall\, w \in W(u). \tag{5.4}$$

The necessary and sufficient conditions for income inequality to decrease, whatever the distribution of productivities, when preferences change are summarised in Table 5.2. Taken together, Tables 5.1 and 5.2 make it clear that the derivative of the consumption function with respect to the logarithm of productivity is the key variable that determines the direction

Table 5.2: Different preferences and identical distributions of productivities

$C(\mathbf{w};\tilde{u}) \geq_{AL} C(\mathbf{w};\hat{u})$		
Consumption Functions	Utility Function $\tilde{u}(c,\ell)$	Utility Function $\hat{u}(c,\ell)$
$\xi(C,w;\tilde{u}) \leqslant \xi(C,w;\hat{u})$	$u^{(2)}(c,\ell) = c - e^{\ell}$	$u^{(5)}(c,\ell) = 2\sqrt{c} - \ell$
	$u^{(12)}(c,\ell) = -(\ell+3)e^{-\frac{c-2}{\ell+3}-1}$	$u^{(13)}(c,\ell) = -(\ell+3)e^{-\frac{c-4}{\ell+3}-1}$

of the change in income inequality when, *either* the distributions of productivity, *or* preferences are modified. The general case where changes affect simultaneously the preferences and the distributions of productivity is easily dealt with. We skip the presentation of the necessary and sufficient conditions for inequality reduction in consumption which are *mutatis mutandis* similar to those obtained in the previous section.

It must be noted that the conditions that guarantee that income inequality – as measured by the relative Lorenz criterion and by its absolute counterpart – decreases as the result of suitable transformations of the distribution of productivities are not independent. Suppose that $\xi(C,w;u)$ is non-increasing, which upon using the fact that $C(w;u)$ is increasing and positive, is equivalent to

$$\eta(C',w;u) \leqslant -1, \; \forall\, w \in W(u). \tag{5.5}$$

Clearly (5.5) implies (4.6), from which we deduce that $\eta(C,w;u)$ is non-increasing in w. In other words, if preferences have the property that absolute income inequality decreases when the agents' productivities become less dispersed and more efficiently allocated, then they also guarantee that relative inequality decreases. Similarly, if $\eta(C,w;u)$ is non-decreasing in w, then so is $\xi(C,w;u)$: the preferences that secure a decrease in relative income inequality when the agents' productivities become less dispersed and less efficiently allocated also ensure that absolute inequality decreases.

6. Quasi-Linear Preferences

So far we have identified the properties of the consumption function that guarantee that inequality in consumption decreases as a result of particular transformations of the distribution of productivities. We have also specified the changes in the consumption function that imply more equal consumptions when individual productivities are fixed and when they are less dispersed. We have also provided evidence by means of examples that there exist preference orderings that generate consumption functions with the desired properties in our artisan economy. However, neither our theoretical results nor our examples provide information on what the underlying preference orders look like. While the consumption function uniquely represents the preference ordering in our model, it may be impossible in practice to derive an explicit representation of this ordering. A first difficulty originates in the fact that it is not always possible to obtain the expenditure function in a closed form starting from the demand system. A second problem is that, even if one succeeded in deriving the expenditure function and in recovering the indirect utility function, then it is difficult to interpret the indifference curves in the *prices-income* space and relate these with the standard indifference curves in the *labour-consumption* space (see however Blackorby, Primont, and Russell (1978)).

Here we restrict our attention to quasi-linear preferences and we look for the implications for the shape of these preferences of the conditions we derived in Sections 4 and 5. Although they are routinely used in many areas, quasi-linear preferences are admittedly restrictive and one may therefore be skeptical about their use here. However, inspection of our examples indicates that quasi-linear preferences are sufficiently flexible to encompass all of the possible situations that we are interested in. Therefore, there is no loss of generality – at least as far as the questions addressed in this paper are concerned – to restrict attention to such preferences. Furthermore, it is always possible in this particular case to solve explicitly the agents' optimisation problems and to derive the explicit form of the consumption function.

Suppose that preferences can be represented by $u(c,\ell) = v(c) - \ell$, where the consumption utility function $v(c)$ is increasing and strictly concave. Maximising $u(c,\ell)$ under the budget constraint $c = w\ell$, we get the necessary and sufficient first order condition $v'(c) = 1/w$, from which we derive the consumption function

$$c = {v'}^{-1}\left(\frac{1}{w}\right) =: C(w;u). \tag{6.1}$$

We consider successively the restrictions to be imposed on the consumption utility function that ensure that income – relative and absolute – inequality decreases.

Relative inequality Using (6.1) and the fact that $w = 1/v'(c)$, the consumption elasticity can be rewritten as

$$\eta(C,w;u) := \frac{C'(w;u)\,w}{C(w;u)} = \frac{1}{-\dfrac{v''(c)\,c}{v'(c)}} = \frac{1}{RRAV(c,u)}, \tag{6.2}$$

where $RRAV(c,v)$ is the relative risk aversion of v evaluated at c. Making use of the fact that $dc/dw > 0$, it follows from (6.2) that the consumption elasticity $\eta(C,w;u)$ is increasing, constant or decreasing in w if and only if the relative risk aversion $RRAV(c,v)$ is decreasing, constant or increasing in c. Denoting by $RRAV_c(c,v)$ the derivative of $RRAV(c,v)$, we obtain

$$RRAV_c(c,v) := -\frac{v'(c)\,[v'''(c)\,c + v''(c)] - v''(c)^2\,c}{v'(c)^2} = -\frac{v'''(c)\,c + v''(c)}{v'(c)} + \frac{v''(c)^2\,c}{v'(c)^2}, \tag{6.3}$$

which, making use of the fact that $v''(c) < 0$ by assumption, implies that

$$RRAV_c(c,v) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \text{ if and only if } RPRU(c,v) - RRAV(c,v) - 1 \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0, \tag{6.4}$$

where $RPRU(c,v) := -v'''(c)\,c/v''(c)$ is the relative prudence of v at c. Table 6.1 provides a summary indicating the connections between the properties of the utility function and those of the corresponding consumption function. Suppose next that there exists a differentiable

Table 6.1: Consumption relative inequality and the properties of the utility function

	Consumption Function Elasticity	Consumption Utility Function
A. Changes in the distribution of productivities		
$\mathbf{w}^* \geq_{LERD} \mathbf{w}^\circ \Longrightarrow C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)$	$\eta_w(C,w;u) \geqslant 0$	$-\dfrac{v'''(c)\,c}{v''(c)} - 1 \geqslant -\dfrac{v''(c)\,c}{v'(c)}$
$\mathbf{w}^* \geq_{MERD} \mathbf{w}^\circ \Longrightarrow C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)$	$\eta_w(C,w;u) \leqslant 0$	$-\dfrac{v'''(c)\,c}{v''(c)} - 1 \leqslant -\dfrac{v''(c)\,c}{v'(c)}$
B. Changes in preferences		
$C(\mathbf{w};\tilde{u}) \geq_{RL} C(\mathbf{w};\hat{u})$	$\eta(C,w;\tilde{u}) \leqslant \eta(C,w;\hat{u})$	$-\dfrac{\tilde{v}''(c)\,c}{\tilde{v}'(c)} \leqslant -\dfrac{\hat{v}''(c)\,c}{\hat{v}'(c)}$

and increasing function $H(w)$ with positive values such that

$$\eta_w(C,w;\tilde{u}) \leqslant \eta_w(H,w) \leqslant \eta_w(C,w;\hat{u}),\ \forall\, w \in W(\tilde{u}) \cap W(\hat{u}). \tag{6.5}$$

Let $c = H(w)$ and consider the function $v\colon W(\tilde{u}) \cap W(\hat{u}) \to \mathbb{R}$ defined by

$$v(c) = \int_0^c \frac{1}{H^{-1}(s)}\, ds. \tag{6.6}$$

We note that $v(c)$ is increasing since $v'(c) = 1/H^{-1}(c) > 0$ and since $w > 0$ by assumption. Furthermore

$$v''(c) = -\frac{1}{[H^{-1}(c)]^2\, H'(w)} < 0, \tag{6.7}$$

since $H(w)$ is increasing. Finally, maximising $u(c,\ell) = v(c) - \ell$ under the budget constraint $c = w\ell$, we get the first order condition

$$v'(c) = \frac{1}{w} = \frac{1}{H^{-1}(c)}, \tag{6.8}$$

which proves that $H(w)$ is the consumption function generated by the utility function $u(c,\ell) = v(c) - \ell$. Then, one can immediately derive the restrictions on the consumption utility function corresponding to the conditions that the elasticity of H is non-decreasing or non-increasing.

Absolute inequality Using (6.1) and the fact that $w = 1/v'(c)$ again, the derivative in the logarithm of the consumption function can be rewritten as

$$\xi(C,w;u) := C'(w;u)\,w = \frac{1}{-\dfrac{v''(c)}{v'(c)}} = \frac{1}{ARAV(c,v)}. \tag{6.9}$$

where $ARAV(c,v)$ is the absolute risk aversion of v evaluated at c. Again, since $dc/dw > 0$, we deduce from (6.9) that the derivative in the logarithm $\xi(C,w;u)$ is increasing, constant or decreasing in w if and only if the absolute risk aversion $ARAV(c,v)$ is decreasing, constant or increasing in c. Letting by $ARAV_c(c,v)$ stand for the derivative of $ARAV(c,v)$, we obtain

$$ARAV_c(c,v) := -\frac{v'(c)\,v'''(c) - v''(c)^2}{v'(c)^2} = -\frac{v''(c)}{v'(c)}\left[\frac{v'''(c)}{v''(c)} - \frac{v''(c)}{v'(c)}\right], \tag{6.10}$$

which, making use of the fact that $v''(c) < 0$ by assumption, implies that

$$ARAV_c(c,v) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \text{ if and only if } APRU(c,v) - ARAV(c,v) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0, \tag{6.11}$$

where $APRU(c,v) := -v'''(c)/v''(c)$ is the absolute prudence of v at c (see, e.g., Eeckhoudt, Gollier, and Schlesinger (1996)). We present in Table 6.2 the properties of the utility function that guarantee that absolute inequality in consumption decreases when productivities are less dispersed and when tastes vary. Suppose next that there exists a differentiable and increasing

Table 6.2: Consumption absolute inequality and the properties of the utility function

	Consumption Derivative in the Logarithm	Consumption Utility Function
A. Changes in the distribution of productivities		
$\mathbf{w}^* \geq_{LERD} \mathbf{w}^\circ \Longrightarrow C(\mathbf{w}^*;u) \geq_{AL} C(\mathbf{w}^\circ;u)$	$\xi_w(C,w;u) \geqslant 0$	$-\dfrac{v'''(c)}{v''(c)} \geqslant -\dfrac{v''(c)\,c}{v'(c)}$
$\mathbf{w}^* \geq_{MERD} \mathbf{w}^\circ \Longrightarrow C(\mathbf{w}^*;u) \geq_{AL} C(\mathbf{w}^\circ;u)$	$\xi_w(C,w;u) \leqslant 0$	$-\dfrac{v'''(c)}{v''(c)} \leqslant -\dfrac{v''(c)\,c}{v'(c)}$
B. Changes in preferences		
$C(\mathbf{w};\tilde{u}) \geq_{AL} C(\mathbf{w};\hat{u})$	$\xi(C,w;\tilde{u}) \leqslant \xi(C,w;\hat{u})$	$-\dfrac{\tilde{v}''(c)}{\tilde{v}'(c)} \leqslant -\dfrac{\hat{v}''(c)}{\hat{v}'(c)}$

function $H(w)$ with positive values such that

$$\xi_w(C,w;\tilde{u}) \leqslant \xi_w(H,w) \leqslant \xi_w(C,w;\hat{u}),\ \forall\, w \in W(\tilde{u}) \cap W(\hat{u}). \tag{6.12}$$

Here again consider the function $v : W(\tilde{u}) \cap W(\hat{u}) \to \mathbb{R}$ defined by (6.6): it is increasing and concave. By construction $H(w)$ solves the optimisation problem of an agent with utility function $u(c,\ell) = v(c) - \ell$ and productivity w. Then, one immediately derives the restrictions on the consumption utility function corresponding to the conditions that the derivative in the logarithm of $H(w)$ is non-decreasing or non-increasing.

By adapting in an appropriate way the previous arguments, one derives analogous results when the preferences are linear in consumption. Consider the utility function $u(c,\ell) = c - \psi(\ell)$, where the disutility function of labour $\psi(\ell)$ is positive, increasing and strictly convex. We note in passing that the corresponding labour supply function $L(w,u)$ is always increasing with productivity: this follows from the assumption that $\psi'(\ell) > 0$ and the Spence-Mirrlees condition. Making use of a little of algebra, we obtain the consumption elasticity:

$$\eta(C,w;u) = \frac{\psi'(\ell)}{\psi''(\ell)\,\ell} + 1 = \frac{-1}{RRAV(\psi,\ell)} + 1, \tag{6.13}$$

where $RRAV(\psi,\ell)$ is the relative risk aversion of ψ evaluated at ℓ. Because $d\ell/dw > 0$, it follows that, for $\eta(C,w;u)$ to increase (resp. decrease) with w, it is necessary and sufficient that $RRAV(\psi,\ell)$ increases (resp. decreases) with ℓ. Things are less transparent when one is interested in the reduction of absolute inequality. In this case the consumption derivative in the logarithm plays a determining role and we obtain:

$$\xi(C,w;u) = \left[\frac{\psi'(\ell)}{\psi''(\ell)} + \ell\right]\psi'(\ell) = \left[\frac{-1}{ARAV(\psi,\ell)} + \ell\right]\psi'(\ell), \tag{6.14}$$

where $ARAV(\psi,\ell)$ is the absolute risk aversion of the disutility of working time. Hence, for $\xi(C,w;u)$ to increase (resp. decrease) with w, it is necessary and sufficient that the left hand side of (6.14) is increasing (resp. decreasing) in working time, a condition whose implications for the structure of preferences are not clear. It is a simple matter to derive the analogues of Tables 6.1 and 6.2 when the preferences are separable and linear in consumption, and we leave this exercise to the reader. We provide in Table A.2 examples of utility functions linear in working time whose consumption elasticities and consumption derivatives in the logarithm are, *either* increasing, *or* decreasing, *or* constant, while Table A.3 gives examples of utility functions linear in consumption whose consumption elasticities have the same properties.

7. Discussion and Concluding Remarks

The paper addressed the general question to know how preferences and productivities interact in the determination of the distribution of income among the society's members. In an artisan economy, where all agents have the same preferences and no exogenous income, the elasticity of the consumption function has proven to be the key variable for determining the impact on relative inequality of changes in both the productivity endowments and the preferences. If the focus is on absolute inequality, then the derivative in the logarithm of the consumption function has been shown to play a similar role. While there is in our model a one-to-one relationship between the consumption function and the preference ordering, it is generally impossible to derive the properties of the preference ordering that constitute the counterparts of the conditions imposed on the elasticity or on the derivative in the logarithm of the consumption function. However, if preferences are quasi-linear, then it is possible to identify the properties of the utility functions that ensure that the restrictions imposed on the elasticity or on the derivative in the logarithm of the consumption function are verified. In this case, the degree of *relative risk aversion* of the consumption utility function is determining in the case of relative inequality, while *absolute risk aversion* is the key parameter when one is interested in absolute inequality.

As we pointed out in the Introduction, the paper does not attempt to explain the upward trend in earnings inequality seen in recent decades. Nevertheless, the small results derived in the paper might contribute – admittedly modestly – to the ongoing debate concerning the factors at the origin of the growing inequalities one might observe across countries and in the course of time. While much of the effort has consisted in searching for the causes of more polarised wages, our results tend to minimise the role played by the wage rates in the increase in earnings inequalities. Or, in the very best case, our results show that particularly restrictive assumptions on preferences have to be made in order that an increase in wage dispersion translates into more unequal earnings. To say things differently, the properties of the labour supply – the counterpart of the consumption function in our model – is likely to play a non-negligible role in the shaping of the earnings profiles.

In addition to the fact that its empirical relevance is limited, the paper can also be attacked on different fronts at the theoretical level. A recurrent critique concerns the emphasis placed on the inequality of earnings, which implicitly raises in turn two questions: (i) Why should we care about inequality rather than, let's say, poverty or welfare? (ii) Why choose earnings as the focus of attention rather than utility for instance? Before addressing these two questions, it must be noted that, had we adopted the approach in terms of *equalities of opportunities*, then we would have restricted attention to wage inequalities. There is indeed no reason to worry about inequalities of earnings to the extent that these simply reflect the fact that the agents have different tastes, in which case the principle of liberty would recommend that we respect their choices. Somehow, we are back to the labour economists' approach that aims at identifying the sources of the inequalities of wages and at quantifying their impacts.

For a number of people – mainly in the US – the fact that the distribution of incomes has become more unequal is not that much of a problem provided that everybody's income has increased at the same time. One can conduct similar investigations as those pursued in the paper by substituting for the – relative or absolute – Lorenz quasi-orderings other criteria that incorporate efficiency considerations in addition to a concern for equality. In this respect, the generalised Lorenz quasi-ordering proposed by Shorrocks (1983) – and its extensions – are immediate candidates for supplementing the standard Lorenz criteria and it is an avenue that is worth exploring. It is expected that one would end up with alternative restrictions to be placed on the consumption function, and a natural question would be to see to which extent these restrictions are compatible with those identified in the paper.[19]

A second objection that can legitimately been made to our approach is that, by focusing exclusively on income and neglecting leisure, it offers a narrow – and possibly distorted – picture of the distribution of well-being in the society. From a social welfare perspective, it would be more appropriate to choose utility as a measure of a person's well-being and therefore to investigate to which extent – and under which conditions – particular modifications of the way productivity is allocated between the agents affect the distribution of utility. The assumption that all agents have the same utility function should in principle allow us to apply the standard Lorenz criteria to the distribution of utilities and look for the properties of the consumption function that ensure that inequalities in utility decreases as the result of less dispersed productivities.[20] As is well-known, the difficulty here is that, contrary to consumption – or, equivalently, to labour supply – the distribution of utilities is not invariant to changes in the utility function chosen for representing the underlying preference ordering.[21]

While the stylised model considered in the paper allowed us to obtain neat conclusions, the question arises to know if these results still hold in a more general setting. Our results suggest a research agenda and the relaxation of everyone of our restrictive assumptions constitutes a potential research avenue. Firstly, we insist on the fact that the assumption that there is no exogenous income is important for our results. Things become more intricate when one relaxes this assumption: for a number of utility functions, the condition of a monotonic consumption elasticity obtained under the assumption that there is no exogenous income no longer holds.[22] This is important because in practice agents differ not only in terms of their

[19] In fact, it can been shown that the conditions identified in Section 5 are necessary and sufficient in order for a uniform proportional progressive transfer of productivities to give rise to a generalised Lorenz improvement in consumption.

[20] Another possibility would be to compare directly the (bi)dimensional distributions of consumption and leisure by means of criteria like those suggested by Atkinson and Bourguignon (1982) as was done, for instance by McCaig and Yatchew (2007). Then, the problem would amount to identifying the properties of the consumption function – or, equivalently, the restrictions to be placed on the preferences – that ensure that less dispersed productivities result in less unequally distributed consumption-leisure bundles.

[21] Consider the utility function $u^{(6)}(c,\ell) := -e^{-c} - \ell$ and the distributions of productivities $\mathbf{w}^\circ = (1.50, 1.55, 2.15, 2.20)$ and $\mathbf{w}^* \simeq (1.725, 1.782, 1.869, 1.913)$. Distribution $\mathbf{w}^*$ is obtained from $\mathbf{w}^\circ$ by means of a uniform proportional progressive transfer plus a small increment to the benefit of the most productive agent, hence $\mathbf{w}^* >_{MERD} \mathbf{w}^\circ$. Because the elasticity of $C(w,u^{(6)})$ is decreasing, we deduce from Proposition 4.3 that $C(\mathbf{w}^*,u^{(6)}) >_{RL} C(\mathbf{w}^\circ,u^{(6)})$. However, the fact that income inequality decreases unambiguously does not prevent inequality of utilities to increase. Indeed, we get $V(\mathbf{w}^\circ,u^{(6)}) >_{RL} V(\mathbf{w}^*,u^{(6)})$, where $V(\mathbf{w}^\circ,u^{(6)})$ and $V(\mathbf{w}^*,u^{(6)})$ represent the distributions of utilities at the equilibrium when the allocations of productivities are respectively $\mathbf{w}^\circ$ and $\mathbf{w}^*$. Now, if we substitute for $u^{(6)}$ the utility function $\tilde{u}^{(6)} := \left(e^{(u^{(6)}+5)}\right)^2$, which is an equally valid representation of the preference ordering, then we obtain the converse ranking, namely $V(\mathbf{w}^*,\tilde{u}^{(6)}) >_{RL} V(\mathbf{w}^\circ,\tilde{u}^{(6)})$.

[22] For linear-in-labour preferences, consumption is independent of exogenous income, which implies that the way the latter is allocated among the population in the artisan economy has no impact on income inequality. When preferences are linear-in-consumption, this is no longer true: depending on the amount of non-labour income, the elasticity of consumption with respect to productivity may be increasing, decreasing or even non-monotonic. Choose the utility function $u^{(2)}(c,\ell) = c - e^{\ell}$ and let $C(w,m;u^{(2)})$ be the corresponding consumption function that depends now on productivity w and exogenous income m. After some algebra, we get $C(w,m;u^{(2)}) = m + w\ln w$ and it can be checked that $\eta(C,w,m;u^{(2)}) = w(1+\ln w)/(m + w\ln w)$ is

productivities but also with respect to their non-labour incomes, and this heterogeneity is likely to be reflected in the way consumption is distributed among the agents at the market equilibrium. Similarly, accounting for the heterogeneity in agents' preferences and the way it interacts with productivities is also worth being investigated. We also acknowledge that the market structure and other institutional arrangements – all of which are absent in our model – are likely to be relevant parameters for explaining the increase in earnings inequalities. Finally, the artisan economy that we considered has the property that the agents cannot improve their situations by transferring parts of their resources. Substituting for our artisan economy a more general production economy where agents have the possibility to trade might well question our results. In particular, the assimilation of wage rates with productivities will no longer hold and preferences are expected to have an impact on the distribution of the wage rates across agents.

8. Proofs of the Results

8.1. Dispersion and Uniform Proportional Progressive Transfers

Proof of Proposition 3.1.

In what follows, we let $\mathbf{v}^*, \mathbf{v}^\circ \in \mathbb{R}^n$ be two distributions such that $v_1^* \leqslant v_2^* \leqslant \cdots \leqslant v_n^*$ and $v_1^\circ \leqslant v_2^\circ \leqslant \cdots \leqslant v_n^\circ$. We will say that $\mathbf{v}^*$ is obtained from $\mathbf{v}^\circ$ by means of a *uniform absolute progressive transfer*, if there exists $\delta, \epsilon > 0$ and two individuals i, j $(1 \leqslant i < j \leqslant n)$ such that:

(8.1a) $$v_h^* = v_h^\circ + \delta, \; \forall\, h \in \{1, \ldots, i\}; \quad v_h^* = v_h^\circ - \epsilon, \; \forall\, h \in \{j, \ldots, n\};$$

(8.1b) $$v_h^* = v_h^\circ, \; \forall\, h \in \{i+1, \ldots, j-1\}; \text{ and}$$

(8.1c) $$i\,\delta = (n - j + 1)\,\epsilon.$$

Clearly, if $\mathbf{w}^* := (w_1^*, \ldots, w_n^*)$ is obtained from $\mathbf{w}^\circ := (w_1^\circ, \ldots, w_n^\circ)$ by means of a *uniform proportional progressive transfer*, then $\ln\mathbf{w}^* := (\ln w_1^*, \ldots, \ln w_n^*)$ is obtained from $\ln\mathbf{w}^\circ := (\ln w_1^\circ, \ldots, \ln w_n^\circ)$ by means of a *uniform absolute progressive transfer*, and conversely. Now, given two distributions $\mathbf{v}^*, \mathbf{v}^\circ \in \mathbb{R}^n$, we say that $\mathbf{v}^*$ *dominates* $\mathbf{v}^\circ$ *in absolute differentials*, which we write $\mathbf{v}^* \geq_{AD} \mathbf{v}^\circ$, if and only if:

(8.2) $$v_i^* - v_i^\circ \geqslant v_j^* - v_j^\circ, \; \forall\, i = 1, 2, \ldots, j-1, \; \forall\, j = 2, 3, \ldots, n.$$

Then, we have the following result due to Chateauneuf, Magdalou, and Moyes (2017) to whom we refer the reader for a proof:

Theorem 8.1. *Let $\mathbf{v}^*, \mathbf{v}^\circ \in \mathbb{R}^n$. The following two statements are equivalent:*

(a) $\mathbf{v}^*$ *is obtained from* $\mathbf{v}^\circ$ *by means of a finite sequence of uniform absolute progressive transfers.*

(b) $\mathbf{v}^* \geq_{AD} \mathbf{v}^\circ$ *and* $\mu(\mathbf{v}^*) = \mu(\mathbf{v}^\circ)$.

Proposition 3.1 follows directly from Theorem 8.1 by letting respectively $v_g^* = \ln w_g^*$ and $v_g^\circ = \ln w_g^\circ$, for all $g = 1, 2, \ldots, n$. □

increasing on the interval $(1, +\infty)$ when $m = 0$ and decreasing otherwise.

8.2. Improvements in Relative Inequality

The following result (see Marshall, Olkin, and Proschan (1967, Theorem 2.4)), which we provide a proof of for completeness, will be used repeatedly in subsequent proofs.

Lemma 8.1. *Let* $\mathbf{c}^*, \mathbf{c}^\circ \in \mathbb{R}^n_{++}$ *such that* $c^*_1 \leqslant c^*_2 \leqslant \cdots \leqslant c^*_n$ *and* $c^\circ_1 \leqslant c^\circ_2 \leqslant \cdots \leqslant c^\circ_n$. *Then* $c^*_1/c^\circ_1 \geqslant c^*_2/c^\circ_2 \geqslant \cdots \geqslant c^*_n/c^\circ_n$ *implies that* $\mathbf{c}^* \geq_{RL} \mathbf{c}^\circ$.

Proof. By definition, for $\mathbf{c}^* \geq_{RL} \mathbf{c}^\circ$, we need that

$$\frac{\sum_{j=1}^k c^*_j}{\sum_{i=1}^k c^*_i + \sum_{i=k+1}^n c^*_i} \geqslant \frac{\sum_{j=1}^k c^\circ_j}{\sum_{i=1}^k c^\circ_i + \sum_{i=k+1}^n c^\circ_i}, \ \forall\, k = 1,2,\ldots,n-1. \tag{8.3}$$

This can be equivalently rewritten as

$$\frac{\sum_{i=1}^k c^\circ_i + \sum_{i=k+1}^n c^\circ_i}{\sum_{j=1}^k c^\circ_j} \geqslant \frac{\sum_{i=1}^k c^*_i + \sum_{i=k+1}^n c^*_i}{\sum_{j=1}^k c^*_j}, \ \forall\, k = 1,2,\ldots,n-1, \tag{8.4}$$

which simplifies to

$$\frac{\sum_{i=k+1}^n c^\circ_i}{\sum_{j=1}^k c^\circ_j} \geqslant \frac{\sum_{i=k+1}^n c^*_i}{\sum_{j=1}^k c^*_j}, \ \forall\, k = 1,2,\ldots,n-1. \tag{8.5}$$

Upon developing (8.5), we obtain

$$\frac{c^\circ_{k+1}}{\sum_{j=1}^k c^\circ_j} + \cdots + \frac{c^\circ_n}{\sum_{j=1}^k c^\circ_j} \geqslant \frac{c^*_{k+1}}{\sum_{j=1}^k c^*_j} + \cdots + \frac{c^*_n}{\sum_{j=1}^k c^*_j}, \ \forall\, k = 1,2,\ldots,n-1. \tag{8.6}$$

For the inequalities (8.6) to be verified, it is sufficient that

$$\frac{\sum_{j=1}^k c^*_j}{c^*_h} \geqslant \frac{\sum_{j=1}^k c^\circ_j}{c^\circ_h}, \ \forall\, h = k+1, k+2, \ldots, n, \ \forall\, k = 1,2,\ldots,n-1. \tag{8.7}$$

Again, a sufficient condition for (8.7) to hold is that $c^*_j/c^*_h \geqslant c^\circ_j/c^\circ_h$, for all $j = 1,2,\ldots,k$, all $h = k+1,\ldots,n$ and all $k = 1,2,\ldots,n-1$, which follows from our assumption that $c^*_i/c^\circ_i \geqslant c^*_{i+1}/c^\circ_{i+1}$, for all $i = 1,2,\ldots,n-1$. □

While all our results involve conditions based on the elasticity of the consumption function, we will dispense with differentiability in the proofs noting that

$$\eta(C, w^\circ; u) \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} \eta(C, w^*; u), \ \forall\, w^\circ < w^* \ (w^\circ, w^* \in W(u)), \tag{8.8}$$

is actually equivalent to

$$\frac{C(\lambda w^\circ; u)}{C(w^\circ; u)} \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} \frac{C(\lambda w^*; u)}{C(w^*; u)}, \ \forall\, \lambda > 1, \ \forall\, w^\circ < w^* \ (w^\circ, w^*, \lambda w^\circ, \lambda w^* \in W(u)), \tag{8.9}$$

in the case of a differentiable consumption function. Finally, the next result borrowed from Aczel (1966, Chapter 3) will prove useful later on.

Lemma 8.2. *The only solution to the functional equation*

$$\frac{C(\lambda w^*;u)}{C(w^*;u)} = \frac{C(\lambda w^\circ;u)}{C(w^\circ;u)},\ \forall\, w^\circ < w^*,\ \forall\, \lambda > 1, \tag{8.10}$$

is given by

$$C(w;u) = \gamma w^\beta\ (\gamma, \beta > 0),\ \forall\, w > 0. \tag{8.11}$$

Proof of Proposition 4.1. Since it is obvious that a constant elasticity is sufficient for conditions (a) to hold, we only prove the necessity part of the proposition. Suppose that $C(\cdot;u)$ is not isoelastic or, equivalently, in virtue of Lemma 8.2, that condition (8.10) is violated, in which case there are two possibilities to be considered.

Case 1: $C(\lambda\hat{w};u)/C(\hat{w};u) < C(\lambda\tilde{w};u)/C(\tilde{w};u)$, for some $\hat{w} < \tilde{w}$ and some $\lambda > 1$. Choose $\mathbf{w}^\circ := (\hat{w},\ldots,\hat{w},\tilde{w})$ and $\mathbf{w}^* := (\lambda\hat{w},\ldots,\lambda\hat{w},\lambda\tilde{w})$, so that $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$. Then, we have

$$\frac{k\,C(\lambda\hat{w};u)}{(n-1)\,C(\lambda\hat{w};u) + C(\lambda\tilde{w};u)} < \frac{k\,C(\hat{w};u)}{(n-1)\,C(\hat{w};u) + C(\tilde{w};u)}, \tag{8.12}$$

for all $k = 1,2,\ldots,n-1$, hence $\neg\,[C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)]$.

Case 2: $C(\lambda\hat{w};u)/C(\hat{w};u) > C(\lambda\tilde{w};u)/C(\tilde{w};u)$, for some $\hat{w} < \tilde{w}$ and some $\lambda > 1$. Choosing now $\mathbf{w}^\circ := (\lambda\hat{w},\ldots,\lambda\hat{w},\lambda\tilde{w})$ and $\mathbf{w}^* := (\hat{w},\ldots,\hat{w},\tilde{w})$, we have $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$. However, we obtain

$$\frac{k\,C(\hat{w};u)}{(n-1)\,C(\hat{w};u) + C(\tilde{w};u)} < \frac{k\,C(\lambda\hat{w};u)}{(n-1)\,C(\lambda\hat{w};u) + C(\lambda\tilde{w};u)}, \tag{8.13}$$

for all $k = 1,2,\ldots,n-1$, and we conclude that $\neg\,[C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)]$. □

Proof of Proposition 4.2. Here again, it is obvious that a constant consumption elasticity is sufficient for conditions (a) to hold.[23] □

Proof of Proposition 4.3.

(b) $\Longrightarrow$ (a). Consider two distributions of productivities $\mathbf{w}^\circ := (w_1^\circ,\ldots,w_n^\circ)$ and $\mathbf{w}^* := (w_1^*,\ldots,w_n^*)$. We have to show that, if

$$\frac{C(\lambda\hat{w};u)}{C(\hat{w};u)} \geqslant \frac{C(\lambda\tilde{w};u)}{C(\tilde{w};u)},\ \forall\, \hat{w} < \tilde{w},\ \forall\, \lambda > 1, \tag{8.14}$$

then $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$ and $\mathbf{w}^* \geq_{ME} \mathbf{w}^\circ$ imply that $C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)$. We have

$$\begin{aligned}
\frac{C(w_i^*;u)}{C(w_i^\circ;u)} &= \frac{C\Big(\Big(\frac{w_i^*}{w_i^\circ}\Big)\, w_i^\circ;u\Big)}{C(w_i^\circ;u)} \geqslant \frac{C\Big(\Big(\frac{w_i^*}{w_i^\circ}\Big)\, w_{i+1}^\circ;u\Big)}{C(w_{i+1}^\circ;u)} \quad \text{(invoking (8.14) since } \mathbf{w}^* \geq_{ME} \mathbf{w}^\circ\text{)} \\
&\geqslant \frac{C\Big(\Big(\frac{w_{i+1}^*}{w_{i+1}^\circ}\Big)\, w_{i+1}^\circ;u\Big)}{C(w_{i+1}^\circ;u)} \quad \text{(since } \mathbf{w}^* \geq_{RD} \mathbf{w}^\circ \text{ and } C(\cdot\,;u) \text{ is non-decreasing)} \\
&= \frac{C(w_{i+1}^*;u)}{C(w_{i+1}^\circ;u)},\ \forall\, i = 1,2,\ldots,n-1,
\end{aligned} \tag{8.15}$$

[23] It is still an open question to know whether the constancy of the consumption elasticity is also necessary for inequality to decrease as the result of a uniform proportional progressive transfer of productivities.

and thanks to Lemma 8.1, we conclude that $C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)$.

(a) $\Longrightarrow$ (b). The proof is analogous to that of Case 1 in Proposition 4.1 and it is omitted. □

Proof of Proposition 4.4.

(b) $\Longrightarrow$ (a). Consider two distributions of productivities $\mathbf{w}^\circ := (w_1^\circ,\dots,w_n^\circ)$ and $\mathbf{w}^* := (w_1^*,\dots,w_n^*)$. We have to show that, if

$$\frac{C(\lambda w^\circ;u)}{C(w^\circ;u)} \leqslant \frac{C(\lambda w^*;u)}{C(w^*;u)},\ \forall\, w^\circ < w^*,\ \forall\, \lambda > 1, \tag{8.16}$$

then $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$ and $\mathbf{w}^* \geq_{LE} \mathbf{w}^\circ$ imply that $C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)$. We have

$$\begin{aligned}\frac{C(w_i^\circ;u)}{C(w_i^*;u)} &= \frac{C\Big(\Big(\frac{w_i^\circ}{w_i^*}\Big)\,w_i^*;u\Big)}{C(w_i^*;u)} \leqslant \frac{C\Big(\Big(\frac{w_i^\circ}{w_i^*}\Big)\,w_{i+1}^*;u\Big)}{C(w_{i+1}^*;u)} \quad \text{(invoking (8.16) since } \mathbf{w}^* \geq_{LE} \mathbf{w}^\circ) \\ &\leqslant \frac{C\Big(\Big(\frac{w_{i+1}^\circ}{w_{i+1}^*}\Big)\,w_{i+1}^*;u\Big)}{C(w_{i+1}^*;u)} \quad \text{(since } \mathbf{w}^* \geq_{RD} \mathbf{w}^\circ \text{ and } C(\cdot\,;u) \text{ is non-decreasing)} \\ &= \frac{C(w_{i+1}^\circ;u)}{C(w_{i+1}^*;u)},\ \forall\, i = 1,2,\dots,n-1,\end{aligned} \tag{8.17}$$

and we deduce from Lemma 8.1 that $C(\mathbf{w}^*;u) \geq_{RL} C(\mathbf{w}^\circ;u)$.

(a) $\Longrightarrow$ (b). The proof is analogous to that of Case 2 in Proposition 4.1 and it is omitted. □

Here again, we find it convenient to express our original conditions involving the elasticities of the consumption function in terms of restrictions that do not involve derivatives. Indeed, we note that, if $C(\cdot\,;u)$ is differentiable, then

$$\eta(C,w;\tilde{u}) \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \eta(C,w;\hat{u}),\ \forall\, w \in W(\tilde{u}) \cap W(\hat{u}), \tag{8.18}$$

is actually equivalent to requiring that

$$\frac{C(\lambda w;\tilde{u})}{C(w;\tilde{u})} \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \frac{C(\lambda w;\hat{u})}{C(w;\hat{u})},\ \forall\, w, \lambda w \in W(\tilde{u}) \cap W(\hat{u}),\ \forall\, \lambda > 1. \tag{8.19}$$

Proof of Proposition 4.5.

(b) $\Longrightarrow$ (a). Given any arbitrary distribution of productivities $\mathbf{w} := (w_1,\dots,w_n) \in \mathscr{W}(\tilde{u}) \cap \mathscr{W}(\hat{u})$, we have to show that, if

$$\frac{C(\lambda w;\tilde{u})}{C(w;\tilde{u})} \leqslant \frac{C(\lambda w;\hat{u})}{C(w;\hat{u})},\ \forall\, w \in W(\tilde{u}) \cap W(\hat{u}),\ \forall\, \lambda > 1, \tag{8.20}$$

then $C(\mathbf{w};\tilde{u}) \geq_{RL} C(\mathbf{w};\hat{u})$. Since by definition $w_{i+1} \geqslant w_i$, for all $i = 1,2,\dots,n-1$, it follows from (8.20) that

$$\frac{C(w_{i+1};\tilde{u})}{C(w_i;\tilde{u})} = \frac{C\Big(\Big(\frac{w_{i+1}}{w_i}\Big)w_i;\tilde{u}\Big)}{C(w_i;\tilde{u})} \leqslant \frac{C\Big(\Big(\frac{w_{i+1}}{w_i}\Big)w_i;\hat{u}\Big)}{C(w_i;\hat{u})} = \frac{C(w_{i+1};\hat{u})}{C(w_i;\hat{u})},\ \forall\, i = 1,2,\dots,n-1, \tag{8.21}$$

and we deduce from Lemma 8.1 that $C(\mathbf{w};\tilde{u}) \geq_{RL} C(\mathbf{w};\hat{u})$.

(a) $\Longrightarrow$ (b). Suppose that $C(\lambda w;\tilde{u})/C(w;\tilde{u}) > C(\lambda w;\hat{u})/C(w;\hat{u})$, for some $\lambda > 1$ and some $w \in W(\tilde{u}) \cap W(\hat{u})$ such that $\lambda w \in W(\tilde{u}) \cap W(\hat{u})$. Choosing $\mathbf{w} := (w, \ldots, w, \lambda w)$, we obtain

$$\frac{k\,C(w;\tilde{u})}{(n-1)\,C(w;\tilde{u}) + C(\lambda w;\tilde{u})} < \frac{k\,C(w;\hat{u})}{(n-1)\,C(w;\hat{u}) + C(\lambda w;\hat{u})}, \tag{8.22}$$

for all $k = 1, 2, \ldots, n-1$, and we conclude that $\neg\,[C(\mathbf{w};\tilde{u}) \geq_{RL} C(\mathbf{w};\hat{u})]$. □

Proof of Proposition 4.6.

(b) $\Longrightarrow$ (a). This follows from combination of Propositions 4.1 and 4.5 (see the discussion in Section 4.5).

(a) $\Longrightarrow$ (b). Let $\mathbf{w}^\circ := (w^\circ, \ldots, w^\circ, \lambda w^\circ)$ and $\mathbf{w}^* := (w^*, \ldots, w^*, \lambda w^*)$, where $\lambda > 1$, w° and w^* are arbitrary but such that $w^\circ, w^*, \lambda w^\circ, \lambda w^* \in W(\tilde{u}) \cap W(\hat{u})$. Clearly $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$. Invoking condition (a), this implies that $C(\mathbf{w}^*;\tilde{u}) \geq_{RL} C(\mathbf{w}^\circ;\hat{u})$, which upon simplifying reduces to

$$\frac{C(\lambda w^\circ;\hat{u})}{C(w^\circ;\hat{u})} \geqslant \frac{C(\lambda w^*;\tilde{u})}{C(w^*;\tilde{u})}\ [>1]. \tag{8.23}$$

Taking the logarithm of both sides of (8.23) and letting λ go to one, we obtain

$$\lim_{\lambda \to 1} \frac{\ln C(\lambda w^\circ;\hat{u}) - \ln C(w^\circ;\hat{u})}{\ln \lambda} \geqslant \lim_{\lambda \to 1} \frac{\ln C(\lambda w^*;\tilde{u}) - \ln C(w^*;\tilde{u})}{\ln \lambda}\ [>0], \tag{8.24}$$

or, equivalently, since $C(\cdot\,;\hat{u})$ and $C(\cdot\,;\tilde{u})$ are differentiable

$$\frac{C'(w^\circ;\hat{u})\,w^\circ}{C(w^\circ;\hat{u})} \geqslant \frac{C'(w^*;\tilde{u})\,w^*}{C(w^*;\tilde{u})}\ [>0], \tag{8.25}$$

which holds for all $w^\circ, w^* \in W(\tilde{u}) \cap W(\hat{u})$. Then, there exists $\beta > 0$ such that

$$\frac{C'(w^\circ;\hat{u})\,w^\circ}{C(w^\circ;\hat{u})} \geqslant \beta \geqslant \frac{C'(w^*;\tilde{u})\,w^*}{C(w^*;\tilde{u})}\ [>0],\ \forall\, w^\circ, w^* \in W(\tilde{u}) \cap W(\hat{u}). \tag{8.26}$$

Defining $H(w) := \gamma w^\beta$ with $\gamma > 0$ arbitrary, we deduce from (8.26) that

$$\frac{C'(w^\circ;\hat{u})\,w^\circ}{C(w^\circ;\hat{u})} \geqslant \frac{H'(w^\circ)\,w^\circ}{H(w^\circ)} = \frac{H'(w^*)\,w^*}{H(w^*)} \geqslant \frac{C'(w^*;\tilde{u})\,w^*}{C(w^*;\tilde{u})}\ [>0], \tag{8.27}$$

which holds for all $w^\circ, w^* \in W(\tilde{u}) \cap W(\hat{u})$. Thus, we have identified a non-decreasing and isoelastic function H such that

$$\frac{C'(w;\hat{u})\,w}{C(w;\hat{u})} \geqslant \frac{H'(w)\,w}{H(w)} \geqslant \frac{C'(w;\tilde{u})\,w}{C(w;\tilde{u})},\ \forall\, w \in W(\tilde{u}) \cap W(\hat{u}), \tag{8.28}$$

which makes the proof complete. □

Proof of Proposition 4.7.

(b) $\Longrightarrow$ (a). This follows from combination of Propositions 4.3 and 4.5 (see the discussion in Section 4.5).

(a) $\Longrightarrow$ (b). Let $\mathbf{w}^\circ := (w^\circ, \dots, w^\circ, \lambda w^\circ)$ and $\mathbf{w}^* := (w^*, \dots, w^*, \lambda w^*)$, where $\lambda > 1$, w° and w^* are arbitrary but such that $w^\circ, w^*, \lambda w^\circ, \lambda w^* \in W(\tilde{u}) \cap W(\hat{u})$. Clearly $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$ and $\mathbf{w}^* \geq_{ME} \mathbf{w}^\circ$. Invoking condition (a), this implies that $C(\mathbf{w}^*; \tilde{u}) \geq_{RL} C(\mathbf{w}^\circ; \hat{u})$. Arguing as in the proof of necessity in Proposition 4.6, we arrive at

$$\text{(8.29)} \qquad \frac{C'(w^\circ; \hat{u})\, w^\circ}{C(w^\circ, \hat{u})} \geqslant \frac{C'(w^*; \tilde{u})\, w^*}{C(w^*; \tilde{u})} \; [> 0],$$

which *now* holds for all $w^\circ \leqslant w^*$ $(w^\circ, w^* \in W(\tilde{u}) \cap W(\hat{u}))$. It follows that

$$\text{(8.30)} \qquad \frac{C'(w^\circ; \hat{u})\, w^\circ}{C(w^\circ; \hat{u})} \geqslant h(w^\circ) := \sup \left\{ \frac{C'(w; \tilde{u})\, w}{C(w; \tilde{u})} \;\middle|\; w \geqslant w^\circ \right\} [> 0],$$

which is true for all $w^\circ \in W(\tilde{u}) \cap W(\hat{u})$). This implies in turn that

$$\text{(8.31)} \qquad \frac{C'(w; \hat{u})\, w}{C(w; \hat{u})} \geqslant h(w) \geqslant \frac{C'(w; \tilde{u})\, w}{C(w; \tilde{u})}, \; \forall\, w \in W(\tilde{u}) \cap W(\hat{u})).$$

We note that by construction h is bounded and non-increasing over $W(\tilde{u}) \cap W(\hat{u})$. Consider the function

$$\text{(8.32)} \qquad H(w) := \exp \int_{\underline{w}}^{w} \frac{h(s)}{s}\, ds, \; \forall\, w \in W(\tilde{u}) \cap W(\hat{u})),$$

where $\underline{w} = \inf\{W(\tilde{u}) \cap W(\hat{u})\}$. The function H has a non-increasing elasticity $h(w) = H'(w)\, w / H(w)$ and it is such that

$$\text{(8.33)} \qquad \frac{C'(w; \hat{u})\, w}{C(w; \hat{u})} \geqslant \frac{H'(w)\, w}{H(w)} \geqslant \frac{C'(w; \tilde{u})\, w}{C(w; \tilde{u})}, \; \forall\, w \in W(\tilde{u}) \cap W(\hat{u}),$$

which makes the proof complete. □

Proof of Proposition 4.8.

(b) $\Longrightarrow$ (a). This follows from combination of Propositions 4.4 and 4.5 (see the discussion in Section 4.5).

(a) $\Longrightarrow$ (b). Similar *mutatis mutandis* to the proof of necessity in Proposition 4.7. Let $\mathbf{w}^\circ := (w^\circ, \dots, w^\circ, \lambda w^\circ)$ and $\mathbf{w}^* := (w^*, \dots, w^*, \lambda w^*)$, where $\lambda > 1$ and $w^\circ \geqslant w^*$ $(w^\circ, w^* \in W(\tilde{u}) \cap W(\hat{u}))$ are arbitrary but such that $\lambda w^\circ, \lambda w^* \in W(\tilde{u}) \cap W(\hat{u})$. Clearly $\mathbf{w}^* \geq_{RD} \mathbf{w}^\circ$ and $\mathbf{w}^* \geq_{LE} \mathbf{w}^\circ$. Invoking condition (a), this implies that $C(\mathbf{w}^*; \tilde{u}) \geq_{RL} C(\mathbf{w}^\circ; \hat{u})$. Arguing as in the proof of necessity in Proposition 4.6, we arrive at

$$\text{(8.34)} \qquad \frac{C'(w^\circ; \hat{u})\, w^\circ}{C(w^\circ; \hat{u})} \geqslant \frac{C'(w^*; \tilde{u})\, w^*}{C(w^*; \tilde{u})} \; [> 0],$$

which *now* holds for all $w^\circ \geqslant w^*$ $(w^\circ, w^* \in W(\tilde{u}) \cap W(\hat{u}))$. It follows that

$$\text{(8.35)} \qquad \inf \left\{ \frac{C'(w; \hat{u})\, w}{C(w; \hat{u})} \;\middle|\; w \geqslant w^* \right\} =: h(w^*) \geqslant \frac{C'(w^*; \tilde{u})\, w^*}{C(w^*; \tilde{u})} \; [> 0],$$

which is true for all $w^* \in W(\tilde{u}) \cap W(\hat{u})$). This implies in turn that

$$\text{(8.36)} \qquad \frac{C'(w; \hat{u})\, w}{C(w; \hat{u})} \geqslant h(w) \geqslant \frac{C'(w; \tilde{u})\, w}{C(w; \tilde{u})}, \; \forall\, w \in W(\tilde{u}) \cap W(\hat{u})).$$

We note that by construction h is bounded and non-decreasing over $W(\tilde{u})\cap W(\hat{u})$. Consider the function

$$H(w) := \exp \int_{\underline{w}}^{w} \frac{h(s)}{s}\, ds,\ \forall\, w \in W(\tilde{u})\cap W(\hat{u})), \tag{8.37}$$

where $\underline{w} = \inf\{W(\tilde{u})\cap W(\hat{u})\}$. The function H has a non-decreasing elasticity $h(w) = H'(w)\,w/H(w)$ and it is such that

$$\frac{C'(w;\hat{u})\,w}{C(w;\hat{u})} \geqslant \frac{H'(w)\,w}{H(w)} \geqslant \frac{C'(w;\tilde{u})\,w}{C(w;\tilde{u})},\ \forall\, w \in W(\tilde{u})\cap W(\hat{u}), \tag{8.38}$$

which makes the proof complete. □

8.3. Improvements in Absolute Inequality

The following result, which indicates a sufficient condition for absolute Lorenz domination, is useful.

Lemma 8.3. *Let $\mathbf{c}^*, \mathbf{c}^\circ \in \mathbb{R}^n$ such that $c_1^* \leqslant c_2^* \leqslant \cdots \leqslant c_n^*$ and $c_1^\circ \leqslant c_2^\circ \leqslant \cdots \leqslant c_n^\circ$. Then $c_1^* - c_1^\circ \geqslant c_2^* - c_2^\circ \geqslant \cdots \geqslant c_n^* - c_n^\circ$ implies that $\mathbf{c}^* \geq_{AL} \mathbf{c}^\circ$.*

Proof. By definition for $\mathbf{x} \geq_{AL} \mathbf{y}$, we need have

$$n \sum_{j=1}^{k} c_j^* - k \sum_{i=1}^{n} c_i^* \geqslant n \sum_{j=1}^{k} c_j^\circ - k \sum_{i=1}^{n} c_i^\circ,\ \forall\, k = 1,2,\ldots,n-1. \tag{8.39}$$

This can be equivalently rewritten as

$$\sum_{j=1}^{k}\left[\sum_{i=1}^{k}\left(c_j^* - c_i^*\right) + \sum_{i=k+1}^{n}\left(c_j^* - c_i^*\right)\right] \geqslant \sum_{j=1}^{k}\left[\sum_{i=1}^{k}\left(c_j^\circ - c_i^\circ\right) + \sum_{i=k+1}^{n}\left(c_j^\circ - c_i^\circ\right)\right], \tag{8.40}$$

for all $k = 1,2,\ldots,n-1$, which reduces to

$$\sum_{j=1}^{k}\sum_{i=k+1}^{n}\left(c_i^* - c_j^*\right) \leqslant \sum_{j=1}^{k}\sum_{i=k+1}^{n}\left(c_i^\circ - c_j^\circ\right),\ \forall\, k = 1,2,\ldots,n-1. \tag{8.41}$$

Upon developing, we obtain

$$\sum_{j=1}^{k}\left(c_{k+1}^* - c_j^*\right) + \cdots + \sum_{j=1}^{k}\left(c_n^* - c_j^*\right) \leqslant \sum_{j=1}^{k}\left(c_{k+1}^\circ - c_j^\circ\right) + \cdots + \sum_{j=1}^{k}\left(c_n^\circ - c_j^\circ\right), \tag{8.42}$$

for all $k = 1,2,\ldots,n-1$. For the $(n-k)$ above inequalities to be verified, it is sufficient that

$$\sum_{j=1}^{k}\left(c_h^* - c_j^*\right) \leqslant \sum_{j=1}^{k}\left(c_h^\circ - c_j^\circ\right),\ \forall\, h = k+1, k+2, \ldots, n,\ \forall\, k = 1,2,\ldots,n-1. \tag{8.43}$$

Again a sufficient condition for (8.43) to hold is that $c_h^* - c_j^* \leqslant c_h^\circ - c_j^\circ$, for all $j = 1,2,\ldots,k$, all $h = k+1,\ldots,n$ and all $k = 1,2,\ldots,n-1$, which follows from our assumption that $c_i^* - c_i^\circ \geqslant c_{i+1}^* - c_{i+1}^\circ$, for all $i = 1,2,\ldots,n-1$. □

The results in Section 5 follow from those in Section 8.2 by substituting for the original consumption function $C(\cdot;u)$ the function $\phi \circ C(\cdot;u)$, with $\phi(s) = \exp(s)$. Indeed, we obtain

$$\eta(\phi \circ C, w; u) = \frac{\phi'(C(w;u))\,C(w;u)}{\phi(C(w;u))}\,\frac{C'(w;u)\,w}{C(w;u)} = C'(w;u)\,w =: \xi(C,w;u), \tag{8.44}$$

for all $w \in W(u)$, which upon substitution in the relevant formula gives the desired results.

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A. Additional Material

A.1. List of the Utility Functions Used in the Paper

We give in Table A.1 the list of the utility functions that we have used at different places in the paper for the sake of illustration. The utility function U^{Stern} is borrowed from Stern (1986).

Table A.1: List of the utility functions used in the paper

Utility Function $u(c,\ell)$	$W(u)$	$\eta_w(C,w;u)$	$\xi_w(C,w;u)$
$u^{(1)}(c,\ell)=c-\dfrac{\ell^2}{2}$	$0<w$	$\eta_w=0$	$\xi_w>0$
$u^{(2)}(c,\ell)=c-e^{\ell}$	$1<w$	$\eta_w<0$	$\xi_w>0$
$u^{(3)}(c,\ell)=\ln c-\dfrac{1}{c}-\ell$	$0<w$	$\eta_w>0$	$\xi_w>0$
$u^{(4)}(c,\ell)=\ln c-\ell$	$0<w$	$\eta_w=0$	$\xi_w>0$
$u^{(5)}(c,\ell)=2\sqrt{c}-\ell$	$0<w$	$\eta_w=0$	$\xi_w>0$
$u^{(6)}(c,\ell)=-e^{-c}-\ell$	$1<w$	$\eta_w<0$	$\xi_w=0$
$u^{(7)}(c,\ell)=-e^{-c}-e^{\ell}$	$1<w$	$\eta_w<0$	$\xi_w>0$
$u^{(8)}(c,\ell)=c-\dfrac{5}{2}\left[\ell e^{\frac{-1}{\ell}}-\displaystyle\int_1^{+\infty}\dfrac{e^{-t\ell}}{t}\,dt\right]$	$0<w<2.5$	$\eta_w>0$	$\xi_w>0$
$u^{(9)}(c,\ell)=\displaystyle\int_0^c\dfrac{1}{H^{-1}(s)}ds-\ell\quad\text{where}\quad H(w)=\dfrac{w^2}{(w^2+(4-w^2))^{\frac{1}{2}}}$	$0<w$	$\eta_w\gtrless 0$	$\xi_w=0$
$u^{(10)}(c,\ell)=c-\dfrac{c^2}{8}-\ell$	$1<w$	$\eta_w<0$	$\xi_w<0$
$u^{\text{Stern}}(c,\ell)=\dfrac{\ell-b}{\chi}\,e^{\frac{\chi(c+a)}{\ell-b}-1}\ \left(a=\dfrac{\rho}{\chi}-\dfrac{\xi}{\chi^2}\,;\ b=\dfrac{\xi}{\chi}\,;\ \chi=-1\,;\ \xi=3\right)$			
$u^{(11)}(c,\ell)=u^{\text{Stern}}(c,\ell)\quad(\rho=0)$	$0<w$	$\eta_w=0$	$\xi_w>0$
$u^{(12)}(c,\ell)=u^{\text{Stern}}(c,\ell)\quad(\rho=-1)$	$\frac{1}{3}<w$	$\eta_w<0$	$\xi_w>0$
$u^{(13)}(c,\ell)=u^{\text{Stern}}(c,\ell)\quad(\rho=+1)$	$0<w$	$\eta_w>0$	$\xi_w>0$
$u^{(14)}(c,\ell)=2\sqrt{c}-\dfrac{\ell^2}{2}$	$0<w$	$\eta_w>0$	$\xi_w>0$

A.2. Details of the Construction of Example 4.1

It is convenient to start by introducing an additional piece of notation. Given the distribution $\mathbf{x}:=(x_1,x_2,\dots,x_n)$ such that $x_1\leqslant x_2\leqslant\cdots\leqslant x_n$, we denote by

$$RL(\mathbf{x}):=\left(RL\left(\frac{1}{n};\mathbf{x}\right),RL\left(\frac{2}{n};\mathbf{x}\right),\dots,RL\left(\frac{n-1}{n};\mathbf{x}\right),1\right),\tag{A.1}$$

the vector of the ordinates of the corresponding relative Lorenz curve, where $RL(k/n;\mathbf{x}) := \sum_{i=1}^{k} x_k/n\mu(\mathbf{x})$, for $k = 1,2,\ldots,n-1$. Consider the utility functions $u^{(1)}$, $u^{(3)}$ and $u^{(9)}$ defined in Table A.1 that all belong to the class $\mathscr{U}$ for appropriate ranges of productivities.

Case 1. Let $n = 4$ and choose $\mathbf{w}^{\circ} = (0.2, 0.5, 3.0, 5.0)$ and $\mathbf{w}^{*} = (0.4, 1.0, 1.5, 2.5)$. We note that $\mathbf{w}^{*}$ is obtained from $\mathbf{w}^{\circ}$ by means of a single uniform proportional progressive transfer.
Assuming that the agents behave in accordance with the preferences represented by the utility function $u^{(1)}$, the allocations of consumption are $C(\mathbf{w}^{*};u^{(1)}) = (0.16, 1.00, 2.25, 6.25)$ and $C(\mathbf{w}^{\circ};u^{(1)}) = (0.04, 0.25, 9.00, 25.00)$. The ordinates of the corresponding relative Lorenz curves are $RL(C(\mathbf{w}^{*};u^{(1)})) = (0.016, 0.120, 0.353, 1.0)$ and $RL(C(\mathbf{w}^{\circ};u^{(1)})) = (0.001, 0.008, 0.270, 1.0)$, and we conclude that $C(\mathbf{w}^{*};u^{(1)}) >_{RL} C(\mathbf{w}^{\circ};u^{(1)})$. Now, suppose the agents adopt the utility function $u^{(9)}$, in which case we get $C(\mathbf{w}^{*};u^{(9)}) = (0.044, 0.316, 0.771, 2.143)$ and $C(\mathbf{w}^{\circ};u^{(9)}) = (0.010, 0.707, 2.846, 4.902)$. The ordinates of the corresponding relative Lorenz curves are $RL(C(\mathbf{w}^{*};u^{(9)})) = (0.013, 0.110, 0.345, 1.0)$ and $RL(C(\mathbf{w}^{\circ};u^{(9)})) = (0.001, 0.010, 0.373, 1.0)$, and we deduce that $C(\mathbf{w}^{*};u^{(1)})$ and $C(\mathbf{w}^{\circ};u^{(1)})$ cannot be ranked by the relative Lorenz quasi-ordering.

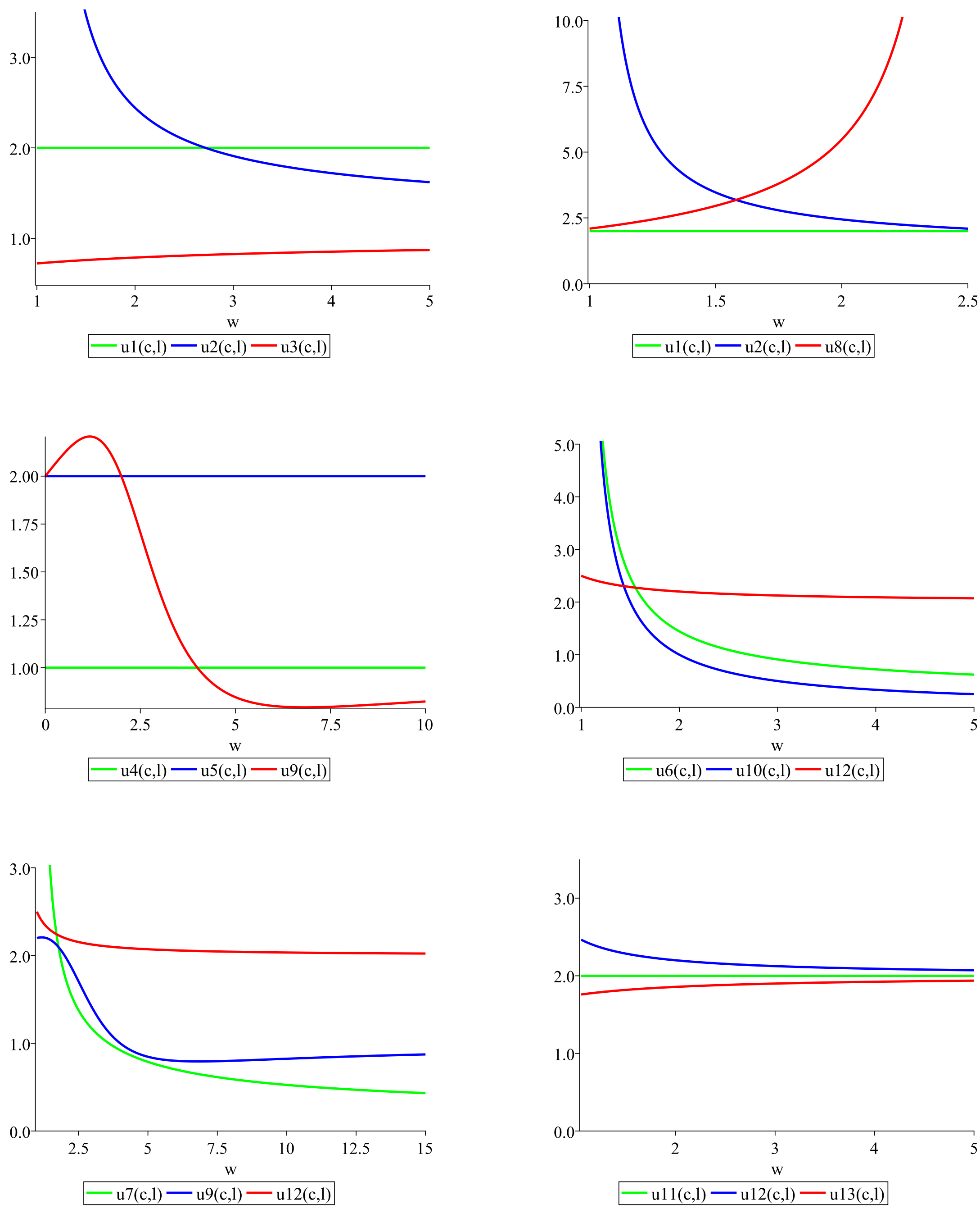
Case 2. Let $n = 2$ and choose $\mathbf{w}^{\circ} = (2.00, 6.00)$ and $\mathbf{w}^{*} = (1.50, 4.00)$. Note that $\mathbf{w}^{*}$ is less dispersed but less efficient than $\mathbf{w}^{\circ}$.
Assuming the utility function $u^{(3)}$, we obtain at the equilibrium $C(\mathbf{w}^{*};u^{(3)}) = (2.186, 4.828)$ and $C(\mathbf{w}^{\circ};u^{(3)}) = (2.732, 6.872)$. The distributions of the corresponding consumption shares are $RL(C(\mathbf{w}^{*};u^{(3)})) = (0.311, 1.0)$ and $RL(C(\mathbf{w}^{\circ};u^{(3)})) = (0.284, 1.0)$, which demonstrates that $C(\mathbf{w}^{*};u^{(3)}) >_{RL} C(\mathbf{w}^{\circ};u^{(3)})$. Now, if the agents adopt the utility function $u^{(9)}$, then the consumption distributions at the equilibrium are $C(\mathbf{w}^{*};u^{(9)}) = (0.771, 4.000)$ and $C(\mathbf{w}^{\circ};u^{(9)}) = (1.414, 5.692)$. The distributions of the consumption shares are $RL(C(\mathbf{w}^{*};u^{(9)})) = (0.161, 1.0)$ and $RL(C(\mathbf{w}^{\circ};u^{(9)})) = (0.199, 1.0)$, which demonstrates that $C(\mathbf{w}^{\circ};u^{(9)}) >_{RL} C(\mathbf{w}^{*};u^{(9)})$.

Case 3. Choose $\mathbf{w}^{\circ} = (0.20, 0.40)$ and $\mathbf{w}^{*} = (0.63, 1.17)$. Note that $\mathbf{w}^{*}$ is less dispersed and more efficient than $\mathbf{w}^{\circ}$.
Assume that the agents have the utility function $u^{(9)}$. Then the consumption distributions at the equilibrium are $C(\mathbf{w}^{*};u^{(9)}) = (0.115, 0.447)$ and $C(\mathbf{w}^{\circ};u^{(9)}) = (0.010, 0.044)$. The distributions of the corresponding consumption shares are $RL(C(\mathbf{w}^{*};u^{(9)})) = (0.205, 1.0)$ and $RL(C(\mathbf{w}^{\circ};u^{(9)})) = (0.192, 1.0)$, which implies that $C(\mathbf{w}^{*};u^{(9)}) >_{RL} C(\mathbf{w}^{\circ};u^{(9)})$. Substituting the utility function $u^{(3)}$ for $u^{(9)}$, we obtain at the equilibrium $C(\mathbf{w}^{*};u^{(3)}) = (1.168, 1.814)$ and $C(\mathbf{w}^{\circ};u^{(3)}) = (0.558, 0.863)$. The distributions of the corresponding consumption shares are $RL(C(\mathbf{w}^{*};u^{(3)})) = (0.391, 1.0)$ and $RL(C(\mathbf{w}^{\circ};u^{(3)})) = (0.392, 1.0)$, and we conclude that $C(\mathbf{w}^{\circ};u^{(3)}) >_{RL} C(\mathbf{w}^{*};u^{(3)})$.

A.3. Consumption Elasticities and Consumption Derivatives in the Logarithms

We consider here a selection of utility functions for which we have plotted the consumption elasticities and the consumption derivatives in the logarithm for admissible ranges of productivities. Figure A.1 provides instances of different situations as far as the elasticity of the consumption function is concerned. Considering first the impact on consumption inequality of less dispersed productivities, the utility functions $u^{(1)}$, $u^{(4)}$, $u^{(5)}$, and $u^{(11)}$ ensure a reduction in consumption inequality (Proposition 4.1). There is no need of additional information

Figure A.1: Consumption elasticities for a selection of utility functions

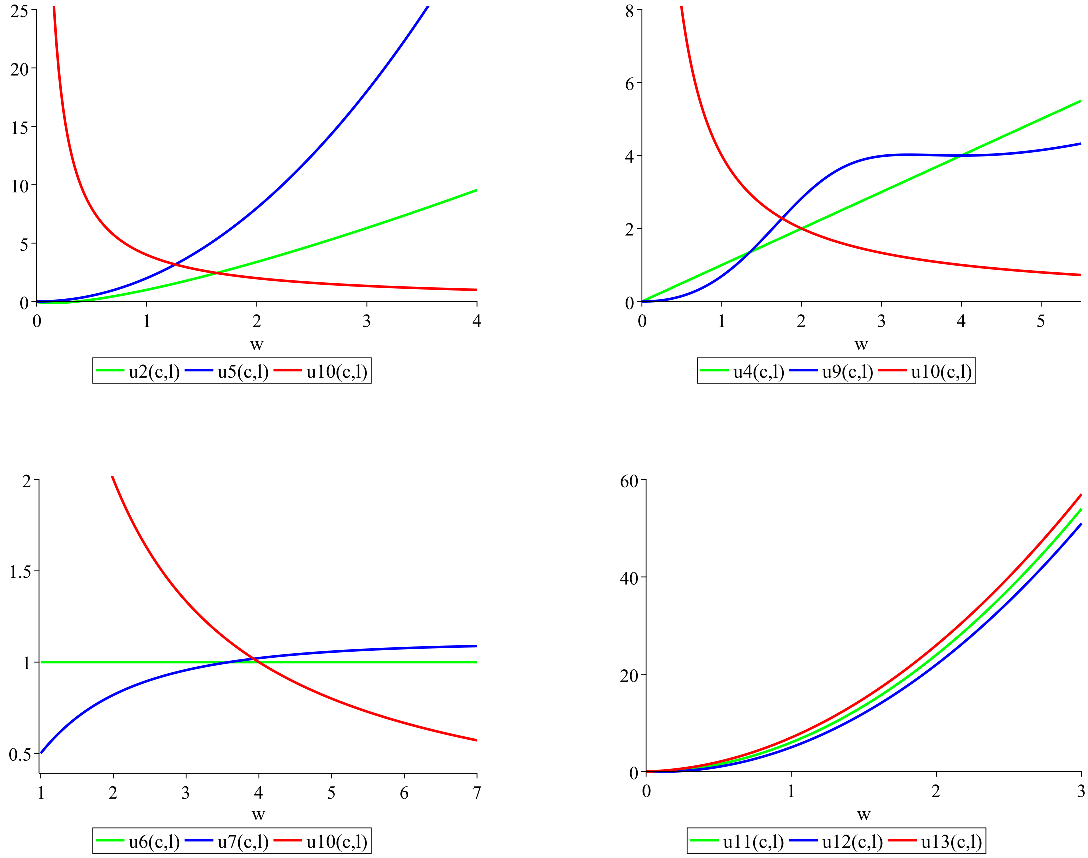


about the distributions of productivities for this result to hold: for instance, it is immaterial whether the reduction of dispersion is accompanied by an increase or a decrease in productive efficiency. The utility functions $u^{(2)}$, $u^{(6)}$, $u^{(7)}$, $u^{(10)}$, and $u^{(12)}$ generate consumption functions whose elasticities are decreasing, which implies that consumption inequality decreases when the distribution of productivities is more concentrated and more efficient (Proposition 4.3), while the utility functions $u^{(3)}$, $u^{(8)}$, and $u^{(13)}$ ensure the same result but when less dispersion in productivity goes along with less efficiency (Proposition 4.4). On the other hand, the utility function $u^{(9)}$ generates a consumption function whose elasticity is non-monotone. It follows that, by choosing distributions of productivities within appropriate ranges, one can obtain everything: the relative Lorenz curves of the corresponding distributions of consumption can move upwards, downwards, or they can intersect. Regarding now the impact on consumption inequality of a change in the preferences, Proposition 4.5 tells us that a switch from $u^{(2)}$ to $u^{(3)}$ $(1 < w)$, from $u^{(8)}$ to $u^{(1)}$ $(0 < w < 2.5)$, from $u^{(5)}$ to $u^{(4)}$ $(0 < w)$, from $u^{(6)}$ to $u^{(10)}$ $(1 < w)$, from $u^{(12)}$ to $u^{(9)}$ $(1/3 < w)$, from $u^{(12)}$ to $u^{(11)}$ $(1/3 < w)$, or from $u^{(11)}$ to $u^{(13)}$ $(0 < w)$ results in an unambiguous reduction of consumption inequality, whatever the distribution of productivities. In all the other cases, the consumption elasticity functions intersect and it is therefore possible to find an inequality index for which the substitution of one utility function for another gives rise to an increase in consumption inequality.

Turning to the joint impact of a modification of the distribution of productivities and of changes in preferences, it can be seen that it is always possible to find increasing functions $f(w)$ and $g(w)$ such that $\eta(C, w; u^{(10)}) < f(w) < \eta(C, w; u^{(6)})$ $(1 < w)$ and $\eta(C, w; u^{(9)}) < g(w) < \eta(C, w; u^{(12)})$ $(1/3 < w)$. It follows that a reduction in productivity dispersion, which degrades efficiency, always smooths the distribution of consumption when the agents substitute the utility function $u^{(10)}$ for the utility function $u^{(6)}$ or the utility function $u^{(12)}$ for the utility function $u^{(12)}$ (Proposition 4.8). Analogously, it is possible to find a decreasing function $h(w)$ such that $\eta(C, w; u^{(11)}) < h(w) < \eta(C, w; u^{(13)})$ $(0 < w)$ which guarantees that consumption inequality always decreases as the result of less dispersed but more efficiently distributed productivities when the agents substitute the utility function $u^{(11)}$ for the utility function $u^{(13)}$ (Proposition 4.7). Finally, we observe that the elasticities of $u^{(3)}$ and $u^{(2)}$, as well as those of $u^{(4)}$ and $u^{(5)}$, can be separated by a constant function, which means that less dispersed productivities – be they more efficiently or less efficiently distributed – will always generate a reduction of consumption inequality when this modification is accompanied by a shift from $u^{(2)}$ to $u^{(3)}$ and from $u^{(5)}$ to $u^{(4)}$ (Proposition 4.6).

Figure A.2 provides similar instances when one is interested in the impact on absolute inequality of changes in preferences and/or modifications of the distributions of productivities. In the absence of additional information about changes in productive efficiency, $u^{(6)}$ is the only utility function that ensures that less dispersed productivities give rise to more equally distributed consumptions: the derivative in the logarithm of the corresponding consumption function is constant. Similarly, $u^{(10)}$ is the only utility function that generates a consumption function whose derivative in the logarithm is decreasing, which guarantees in turn that less dispersed and more efficiently distributed productivities always translate into more equally distributed consumptions. Choosing anyone of the other utility functions, with the exception of $u^{(9)}$, ensures that inequality decreases as the result of less dispersed and less efficiently distributed productivities. The derivative in the logarithm of the consumption function corresponding to of $u^{(9)}$ is non-monotone. This implies that the impact on consumption absolute inequality of less dispersed productivities is ambiguous: It is possible, by choosing distributions of productivities within appropriate ranges, to make the absolute Lorenz curves of the corresponding distributions of consumption move upwards, downwards, or intersect. The only cases where consumption absolute inequality decreases as the result of a change in the preferences are when $u^{(2)}$ is substituted for $u^{(5)}$ $(1 < w)$, $u^{(12)}$ for $u^{(11)}$ $(1/3 < w)$, $u^{(11)}$ for $u^{(13)}$

Figure A.2: Consumption derivatives in the logarithm for a selection of utility functions



$(0 < w)$, as well as $u^{(12)}$ for $u^{(13)}$ $(1/3 < w)$. In all the other cases, the graphs of the derivatives in the logarithm of the consumption functions intersect, which makes it impossible to predict what will be the impact on consumption absolute inequality of the substitution of one utility function for another in the absence of additional information about the distribution of productivities. Considering the effect of simultaneous changes in preferences and in the way productivities are allocated among the agents, the only instances where it is possible to get a conclusive verdict are given by the utility functions $u^{(2)}$ and $u^{(5)}$ on the one hand, and the utility functions $u^{(11)}$, $u^{(12)}$ and $u^{(13)}$ on the other hand. By substituting $u^{(2)}$ for $u^{(5)}$, $u^{(11)}$ for $u^{(13)}$, $u^{(12)}$ for $u^{(11)}$, or $u^{(12)}$ for $u^{(13)}$, while reducing dispersion and decreasing productive efficiency at the same time, one gets an unambiguous decrease of consumption absolute inequality.

A.4. Uniform Proportional Progressive Transfers and Relative Differentials Domination

Proposition 4.2 indicates that a constant consumption elasticity is sufficient for inequality to decrease as the result of uniform proportional progressive transfers of productivities. However, it is not clear whether this property of the consumption function is also necessary to achieve greater equality in these particular circumstances. The following result confirms that a constant consumption elasticity is both necessary and sufficient in order for uniform proportional progressive transfers to generate less dispersed – as measured by the relative differentials quasi-ordering – consumption levels. We insist on the fact that, in order to prove this result, additional restrictions are needed, namely: $n \geqslant 4$ and $w_- < w < w_+$, where $0 < w_- < w_+ < +\infty$. Given the utility function $u \in \mathscr{U}$, we let $W^*(u) := W(u) \cap (w_-, w_+)$ and we indicate by $\mathscr{W}^*(u)$ the set of allocations of productivities $\mathbf{w} := (w_1, \ldots, w_n)$ such that $w_i \in W^*(u)$, for all $i = 1, 2, \ldots, n$.

Proposition A.1. *Let $u \in \mathscr{U}$ and $n \geqslant 4$. The following two statements are equivalent:*

(a) *For all $\mathbf{w}^*, \mathbf{w}^\circ \in \mathscr{W}^*(u)$; $C(\mathbf{w}^*; u) \geq_{RD} C(\mathbf{w}^\circ; u)$ whenever $\mathbf{w}^*$ is obtained from $\mathbf{w}^\circ$ by means of uniform proportional progressive transfers.*

(b) *$\eta(C, w; u)$ is constant in w, for all $w \in W^*(u)$.*

Proof. We find convenient to argue indirectly and, to this aim, we define $f(w) := C(w; u)$, $v := \ln w$ and $\phi(v) := \ln \circ f \circ \exp(v)$, and we note that by construction $\phi(v)$ is continuous and increasing.

(b) $\Longrightarrow$ (a). With the above conventions, the proof amounts to showing that, if $\phi(v) = \alpha + \beta v$ with $\beta > 0$, then $\phi(\mathbf{v}^*) \geq_{AD} \phi(\mathbf{v}^\circ)$ whenever $\mathbf{v}^*$ is obtained from $\mathbf{v}^\circ$ by means of a finite sequence of uniform absolute progressive transfers. Suppose without loss of generality that $\mathbf{v}^* := (v_1^*, \ldots, v_n^*)$ is obtained from $\mathbf{v}^\circ := (v_1^\circ, \ldots, v_n^\circ)$ by means of a single uniform absolute progressive transfer so that:

$$\text{(A.2a)} \qquad \mathbf{v}^*: v_1^\circ + \delta \leqslant \cdots \leqslant v_i^\circ + \delta \leqslant v_{i+1}^\circ \leqslant \cdots \leqslant v_{j-1}^\circ \leqslant v_j^\circ - \epsilon \leqslant \cdots \leqslant v_n^\circ - \epsilon \text{ and}$$

$$\text{(A.2b)} \qquad \mathbf{v}^\circ: \ v_1^\circ \leqslant \cdots \leqslant \ v_i^\circ \leqslant v_{i+1}^\circ \leqslant \cdots \leqslant v_{j-1}^\circ \leqslant v_j^\circ \leqslant \cdots \leqslant v_n^\circ,$$

where $i\delta = (n - j + 1)\epsilon$. We know that $\phi(v) = \alpha + \beta v$ is equivalent to $\phi(\hat{v} + \Delta) - \phi(\hat{v}) = \phi(\tilde{v} + \Delta) - \phi(\tilde{v})$, for all $\Delta > 0$ and all $\hat{v}, \tilde{v} \in \ln W^*(u)$ such that $\hat{v} + \Delta, \tilde{v} + \Delta \in \ln W^*(u)$ (see Aczel (1966, Chapter 2)). This implies that:

$$\text{(A.3a)} \qquad \phi(v_1^\circ + \delta) - \phi(v_1^\circ) = \cdots = \phi(v_i^\circ + \delta) - \phi(v_i^\circ) = \beta\delta;$$

$$\text{(A.3b)} \qquad \phi(v_{i+1}^\circ) - \phi(v_{i+1}^\circ) = \cdots = \phi(v_{j-1}^\circ) - \phi(v_{j-1}^\circ) = 0; \text{ and}$$

$$\text{(A.3c)} \qquad \phi(v_j^\circ - \epsilon) - \phi(v_j^\circ) = \cdots = \phi(v_n^\circ - \epsilon) - \phi(v_n^\circ) = -\beta\epsilon.$$

For condition (a) to be fulfilled, it suffices that $\beta\delta \geqslant 0 \geqslant -\beta\epsilon$. Actually, we have $\beta\delta > 0$, which follows from the fact that $\delta > 0$ by definition of a uniform absolute progressive and the assumption that $\beta > 0$. By definition of a uniform absolute progressive transfer, we have $\epsilon = (i/(n - j + 1))\delta$, which is strictly positive because $1 \leqslant i < j \leqslant n$. Therefore, we have $\beta\delta > 0 > -\beta\epsilon$ and we conclude that $\phi(\mathbf{v}^*) \geq_{AD} \phi(\mathbf{v}^\circ)$, hence $C(\mathbf{w}^*; u) \geq_{RD} C(\mathbf{w}^\circ; u)$.

(a) $\Longrightarrow$ (b). We show that, if $\phi(\mathbf{v}^*) \geq_{AD} \phi(\mathbf{v}^\circ)$ whenever $\mathbf{v}^*$ is obtained from $\mathbf{v}^\circ$ by means of uniform absolute progressive transfers, then $\phi(v) = \alpha + \beta v$ with $\beta > 0$. Let $a := \ln w_-$ and $b := \ln w_+$. Given the interval $V := (a, b)$ and the integer $q \geqslant 3$, consider the 2^q following subintervals:

$$\text{(A.4a)} \qquad V_1 := (a + 0\epsilon, a + 1\epsilon];$$

$$\text{(A.4b)} \qquad V_h := [a + (h-1)\epsilon, a + h\epsilon], \text{ for } h = 2, 3, \ldots, 2^q - 1;$$

(A.4c) $$V_{2^q} := [\,a + (2^q - 1)\,\epsilon, a + 2^q\,\epsilon\,)\,;$$

and note that by definition $\epsilon = (b-a)/2^q$ and $(a,b) = \bigcup_{h=1}^{2^q} V_h$. We proceed in $(2^q - 2)/2 = 2^{q-1} - 1$ successive steps.

$\underline{t=1}$: Interval $[a + 0\,\epsilon, a + 4\,\epsilon]$. Consider the two following distributions:

(A.5a) $$\mathbf{v}^\circ := (a + 0\,\epsilon, a + 1\,\epsilon, a + 2\,\epsilon, \ldots, a + 2\,\epsilon, a + 3\,\epsilon, a + 4\,\epsilon);$$

(A.5b) $$\mathbf{v}^* := (a + 1\,\epsilon, a + 2\,\epsilon, a + 2\,\epsilon, \ldots, a + 2\,\epsilon, a + 2\,\epsilon, a + 3\,\epsilon).$$

Clearly, distribution $\mathbf{v}^*$ is obtained from distribution $\mathbf{v}^\circ$ by means of a uniform absolute progressive transfer. Assume that $\phi(\mathbf{v}^*) \geq_{AD} \phi(\mathbf{v}^\circ)$, which implies that:

(A.6) $$\phi(a+\epsilon) - \phi(a) \geqslant \phi(a+2\epsilon) - \phi(a+\epsilon) \geqslant \phi(a+2\epsilon) - \phi(a+3\epsilon) \geqslant \phi(a+3\epsilon) - \phi(a+4\epsilon),$$

from which we deduce that

(A.7a) $$2\,\phi\,(a+\epsilon) \geqslant \phi(a) + \phi(a+2\epsilon) \text{ and}$$

(A.7b) $$2\,\phi\,(a+3\epsilon) \leqslant \phi(a+2\epsilon) + \phi(a+4\epsilon).$$

$\underline{t=2}$: Interval $[a + 2\,\epsilon, a + 6\,\epsilon]$. Choose now the distributions:

(A.8a) $$\mathbf{v}^\circ := (a + 2\,\epsilon, a + 3\,\epsilon, a + 4\,\epsilon, \ldots, a + 4\,\epsilon, a + 5\,\epsilon, a + 6\,\epsilon);$$

(A.8b) $$\mathbf{v}^* := (a + 3\,\epsilon, a + 4\,\epsilon, a + 4\,\epsilon, \ldots, a + 4\,\epsilon, a + 4\,\epsilon, a + 5\,\epsilon).$$

Again, distribution $\mathbf{v}^*$ is obtained from distribution $\mathbf{v}^\circ$ by means of a uniform absolute progressive transfer and, assuming that $\phi(\mathbf{v}^*) \geq_{AD} \phi(\mathbf{v}^\circ)$, we have:

(A.9) $$\phi(a{+}3\epsilon) - \phi(a{+}2\epsilon) \geqslant \phi(a{+}4\epsilon) - \phi(a{+}3\epsilon) \geqslant \phi(a{+}4\epsilon) - \phi(a{+}5\epsilon) \geqslant \phi(a{+}5\epsilon) - \phi(a{+}6\epsilon),$$

from which we deduce that

(A.10a) $$2\,\phi\,(a+3\epsilon) \geqslant \phi(a+2\epsilon) + \phi(a+4\epsilon) \text{ and}$$

(A.10b) $$2\,\phi\,(a+5\epsilon) \leqslant \phi(a+4\epsilon) + \phi(a+6\epsilon).$$

Upon combining (A.7b) and (A.10a), we obtain:

(A.11) $$2\,\phi\,(a+3\epsilon) = \phi(a+2\epsilon) + \phi(a+4\epsilon).$$

$$\vdots$$

$\underline{t = 2^{q-1} - 2}$: Interval $[a + (2^q - 6)\,\epsilon, a + (2^q - 2)\,\epsilon]$. Consider the distributions:

(A.12a) $$\mathbf{v}^\circ := (a{+}(2^q{-}6)\epsilon, a{+}(2^q{-}5)\epsilon, a{+}(2^q{-}4)\epsilon, \ldots, a{+}(2^q{-}4)\epsilon, a{+}(2^q{-}3)\epsilon, a{+}(2^q{-}2)\epsilon);$$

(A.12b) $$\mathbf{v}^* := (a{+}(2^q{-}5)\epsilon, a{+}(2^q{-}4)\epsilon, a{+}(2^q{-}4)\epsilon, \ldots, a{+}(2^q{-}4)\epsilon, a{+}(2^q{-}4)\epsilon, a{+}(2^q{-}3)\epsilon),$$

so that $\mathbf{v}^\circ$ is transformed into $\mathbf{v}^*$ by means of a uniform absolute progressive transfer. Assuming that $\phi(\mathbf{v}^*) \geq_{AD} \phi(\mathbf{v}^\circ)$, we deduce that

(A.13a) $$2\phi(a+(2^q-5)\epsilon) \geqslant \phi(a+(2^q-6)\epsilon)+\phi(a+(2^q-4)\epsilon);$$

(A.13b) $$2\phi(a+(2^q-3)\epsilon) \leqslant \phi(a+(2^q-4)\epsilon)+\phi(a+(2^q-2)\epsilon).$$

Appealing to the same argument as that used in Step 2, we conclude that:

(A.14) $$2\phi(a+(2^q-5)\epsilon) = \phi(a+(2^q-6)\epsilon)+\phi(a+(2^q-4)\epsilon).$$

$\underline{t=2^{q-1}-1}$: Interval $[a+(2^q-4)\epsilon, a+2^q\epsilon]$. In this last step, we consider the distributions:

(A.15a) $$\mathbf{v}^\circ := (a+(2^q-4)\epsilon, a+(2^q-3)\epsilon, a+(2^q-2)\epsilon, \ldots, a+(2^q-2)\epsilon, a+(2^q-1)\epsilon, a+2^q\epsilon);$$

(A.15b) $$\mathbf{v}^* := (a+(2^q-3)\epsilon, a+(2^q-2)\epsilon, a+(2^q-2)\epsilon, \ldots, a+(2^q-2)\epsilon, a+(2^q-2)\epsilon, a+(2^q-1)\epsilon).$$

Distribution $\mathbf{v}^*$ is obtained from distribution $\mathbf{v}^\circ$ by means of a uniform absolute progressive transfer. Since by assumption $\phi(\mathbf{v}^*) \geq_{AD} \phi(\mathbf{v}^\circ)$, we deduce that:

(A.16a) $$2\phi(a+(2^q-3)\epsilon) \geqslant \phi(a+(2^q-4)\epsilon)+\phi(a+(2^q-2)\epsilon);$$

(A.16b) $$2\phi(a+(2^q-1)\epsilon) \leqslant \phi(a+(2^q-2)\epsilon)+\phi(a+2^q\epsilon).$$

Upon combining (A.13b) and (A.16a), we obtain:

(A.17) $$2\phi(a+(2^q-3)\epsilon) = \phi(a+(2^q-4)\epsilon)+\phi(a+(2^q-2)\epsilon).$$

To summarise, we have shown that

(A.18a) $$2\phi(a+\epsilon) \geqslant \phi(a)+\phi(a+2\epsilon);$$

(A.18b) $$2\phi\Big(a+(2^t-1)\epsilon\Big) = \phi(a+(2^t-2)\epsilon)+\phi(a+2^t\epsilon), \text{ for } t=2,3,\ldots,2^{q-1}-2;$$

(A.18c) $$2\phi(a+(2^q-1)\epsilon) \leqslant \phi(a+(2^q-2)\epsilon)+\phi(a+2^q\epsilon),$$

which holds for all $q \geqslant 3$. By varying q from 3 to infinity, we deduce from (A.18b) that $2\phi((s+r)/2) = \phi(s)+\phi(r)$, for all $s,r \in (a,b)$ with $s<r$. Note in passing that

(A.19a) $$a+2\epsilon = a+\frac{b-a}{2^q} \to a \text{ and}$$

(A.19b) $$a+(2^q-2)\epsilon = a+(2^q-2)\frac{b-a}{2^q} = b-2\frac{b-a}{2^q} \to b,$$

whenever q goes to infinity. Since $\phi = C(\cdot\,;u)$ is continuous, we conclude that ϕ is linear over (a,b) (see Aczel (1966, Chapter 2)). Hence, $\phi(v) = \alpha+\beta v$ ($\beta>0$), for all $v \in (a,b)$. Finally, substituting for v and ϕ their respective definitions, we obtain

(A.20) $$C(w;u) =: f(w) = \exp(\alpha+\beta\ln w) = \gamma w^\beta,$$

where $\gamma := \exp(\alpha)$, which makes the proof complete. □

A.5. Uniform Proportional Progressive Transfers and the Variance of the Logarithms

First, we note that, if $\mathbf{w}^*$ is obtained from $\mathbf{w}^\circ$ by means of a uniform proportional progressive transfer, then $\ln\mathbf{w}^*$ is obtained from $\ln\mathbf{w}^\circ$ by means of a uniform absolute progressive transfer as defined by (8.1), and conversely. The variance of distribution $\mathbf{w} := (w_1, \ldots, w_n)$ is given by

$$V(\mathbf{w}) := \frac{1}{n}\sum_{i=1}^{n}\Big(w_i - \mu(\mathbf{w})\Big)^2 = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \mu(\mathbf{w})^2, \tag{A.21}$$

where $\mu(\mathbf{w})$ is the arithmetic mean of $\mathbf{w}$, while the variance of the logarithms is equal to

$$VL(\mathbf{w}) := \frac{1}{n}\sum_{i=1}^{n}\Big(\ln w_i - \ln\nu(\mathbf{w})\Big)^2 = \frac{1}{n}\sum_{i=1}^{n}\Big(\ln w_i\Big)^2 - \Big(\ln\nu(\mathbf{w})\Big)^2, \tag{A.22}$$

where $\nu(\mathbf{w})$ is the geometric mean of $\mathbf{w}$. Noting that $\ln\nu(\mathbf{w}) = \mu(\ln\mathbf{w})$ makes clear that $VL(\mathbf{w}) = V(\ln\mathbf{w})$. Show that $VL(\mathbf{w}^*) \leqslant VL(\mathbf{w}^\circ)$ whenever $\mathbf{w}^*$ is derived from $\mathbf{w}^\circ$ by means of a uniform proportional progressive transfer amounts to verifying that $V(\ln\mathbf{w}^*) \leqslant V(\ln\mathbf{w}^\circ)$ whenever $\ln\mathbf{w}^*$ is obtained from $\ln\mathbf{w}^\circ$ by means of a uniform absolute progressive transfer. Interestingly, a uniform absolute progressive transfer can always be decomposed into a finite number of (standard) progressive transfers. This can be shown directly, but it is tedious. Another (indirect) argument is to invoke the following facts: (i) if one distribution is obtained from another by means of uniform absolute progressive transfers, then it dominates the latter according to the absolute differentials quasi-ordering – actually the implication goes both ways – (see Theorem 8.1), (ii) dominance in terms of absolute differentials implies Lorenz dominance (see Chateauneuf *et al.* (2017)), and (iii) the Hardy-Littlewood-Polyá theorem tells us, among other things, that if one distribution Lorenz dominates another, then it can always be obtained from the latter by means of a finite number of progressive transfers. Then, it suffices to check that a progressive transfer reduces the variance, which follows from the Hardy-Littlewood-Polyá theorem again and the fact that the function $\phi(s) := s^2$ is convex.

A.6. Quasi-linear Preferences

We have identified in Section 6 the properties of the utility functions that constitute the counterparts of the consumption elasticities and consumption derivatives in the logarithm

Table A.2: Preferences linear in working time

Utility Function	Relative Inequality		Absolute Inequality	
	$\eta_w(C,w;u)$	$RRAV_c(v,c)$	$\xi_w(C,w;u)$	$ARAV_c(v,c)$
$u^{(3)}(c,\ell) = \ln c - \frac{1}{c} - \ell$	>0	<0	>0	<0
$u^{(6)}(c,\ell) = -e^{-c} - \ell$	<0	>0	$=0$	$=0$
$u^{(10)}(c,\ell) = c - \frac{c^2}{8} - \ell$	<0	>0	<0	>0
$u^{(5)}(c,\ell) = 2\sqrt{c} - \ell$	$=0$	$=0$	>0	<0

conditions in the particular case where preferences are, *either* linear in working time, *or* linear in consumption. We give in Table A.2 examples of utility functions linear in working time that have the property to generate consumption elasticities and consumption derivatives in the logarithm that are, *either* increasing, *or* decreasing, *or* constant. Table A.3 provides a similar illustration in the case where the utility functions are linear in consumption. Put together, these two tables should help to convince the reader that restricting attention to quasi-linear preferences is not as restrictive as it might look at first sight.

Table A.3: Preferences linear in consumption

	Relative Inequality		Absolute Inequality	
Utility Function	$\eta_w(C,w;u)$	$RRAV_c(v,c)$	$\xi_w(C,w;u)$	$ARAV_c(v,c)$
$u^{(2)}(c,\ell)=c-e^{\ell}$	<0	<0	>0	>0
$u^{(1)}(c,\ell)=c-\dfrac{\ell^2}{2}$	$=0$	$=0$	>0	>0
$u^{(8)}(c,\ell)=c-\dfrac{5}{2}\left[\ell e^{\frac{-1}{\ell}}-\int_1^{+\infty}\dfrac{e^{-t\ell}}{t}\,dt\right]$	>0	>0	>0	>0

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