Groupe de Recherche en
Économie Théorique et Appliquée

Viability standards and multi-criteria maximin

Pedro Gajardo

Universidad Técnica Federico Santa María,

Valparaíso, Chile

pedro.gajardo@usm.cl

&

Luc Doyen

GREThA, CNRS, UMR 5113, University of Bordeaux

luc.doyen@u-bordeaux.fr

Cahiers du GREThA
n° 2018-04
February

GRETHA UMR CNRS 5113
Université de Bordeaux
Avenue Léon Duguit - 33608 PESSAC - FRANCE
Tel : +33 (0)5.56.84.25.75 - Fax : +33 (0)5.56.84.86.47 - www.gretha.fr

Normes de viabilité et maximin multi-critères

Résumé

Ce document traite des critères et des normes de durabilité. Dans cette perspective, le critère de maximin, en tant que performance la plus élevée pouvant être maintenue au cours du temps, favorise l'équité intergénérationnelle, un enjeu crucial pour la durabilité. Par ailleurs, l'approche du contrôle viable qui met en évidence des trajectoires et des décisions respectant diverses normes et contraintes au cours du temps, fournit des indications majeures sur la durabilité forte. Le présent article aborde les liens entre les approches de maximin et de viabilité dans un contexte multi-critères. Il montre d'abord comment le maximin 'Pareto' peut être caractérisé via les noyaux de viabilité. Un tel résultat permet de déterminer les compromis et/ou les synergies entre les normes économiques et écologiques non substituables qui sous-tendent la durabilité forte. Le deuxième résultat principal de l'article est de proposer des algorithmes de programmation dynamique pour approcher numériquement les valeurs maximales de Pareto maximin, les contrôles et les normes durables. Deux exemples reposant sur la gestion des ressources renouvelables illustrent ces résultats analytiques et numériques.

Mots-clés: Durabilité forte, Maximin, Multi-critères, Viabilité, Programmation dynamique, Ressources renouvelables

Viability standards and multi-criteria maximin

Abstract

This paper deals with sustainability criteria and standards. The maximin criterion, as the highest performance that can be sustained over time, promotes intergenerational equity, a pivotal issue for sustainability. The viable control approach, by investigating trajectories and actions complying over time with various standards and constraints, provides major insights into strong sustainability. The present paper addresses the links between maximin and viability approaches in a multi-criteria context. It first shows how 'Pareto maximin' can be characterized through viability kernels. Such a result makes it possible to determine the trade-offs and/or synergies between non-substitutable economic and ecological standards underlying strong sustainability. The second main result of the paper is to propose algorithms derived from the viability version of dynamic programming to approximate numerically Pareto maximin values, controls and sustainable standards. Two examples relying on renewable resource management illustrate these analytic and numerical findings.

Keywords: Strong Sustainability, Maximin, Multi-criteria, Viability, Dynamic programming, Renewable resources

JEL: Q22-C53

Reference to this paper: GAJARDO Pedro, DOYEN Luc (2018) Viability standards and multi-criteria maximin, *Cahiers du GREThA*, n°2018-04.
http://ideas.repec.org/p/grt/wpegrt/2018-04.html.

GRETHA UMR CNRS 5113
Université de Bordeaux
Avenue Léon Duguit - 33608 PESSAC - FRANCE
Tel : +33 (0)5.56.84.25.75 - Fax : +33 (0)5.56.84.86.47 - www.gretha.fr

1. Introduction

Operationalizing sustainability is a major challenge in terms of objectives, quantitative methods and criteria (Cairns and Long, 2006; Asheim, 2007; Doyen and Martinet, 2012; Fleurbaey, 2015), in particular to address ecological-economic issues. In that respect, the discounted utility approach, which is broadly used in the context of optimal control, is criticized since this criterion may lead to unsustainable economic trajectories mainly because of discounting. The discounted utility criterion indeed neglects long-run utility and is qualified as a 'dictatorship of the present' by Chichilnisky (1996). Accordingly, the studies related to quantitative sustainability have been characterized by the introduction and analysis of other criteria including the maximin (Solow, 1974). This maximin criterion, defined as the highest utility level that can be sustained over time, is well-suited to promote intergenerational equity.

However, several authors question the relevance of optimization methods including the maximin approach to quantify sustainability. In particular, Howarth (1995); Martinet (2011) note that sustainability conditions should be imposed as prior constraints on the maximization of any social welfare function. Thus, the use of biological, ecological or physical targets and constraints is advocated to address the sustainability issue, due to its long-term perspective. This approach is the one favored to address the climate change issue in practice, as illustrated by the Kyoto protocol, which defines physical targets in quantitative terms. Similarly, in biodiversity conservation or management, ecological constraints are usually preferred to utility or monetary evaluations such as the Aichi Biodiversity Targets formulated by the Convention on Biological Diversity (CBD. In this context, reference points not to exceed these bioeconomic indicators stand for sustainable management objectives. The Tolerable Windows Approach (TWA), as described in Bruckner et al. (1999), also relies on safe boundaries and feasibility regions. To meet the challenge of sustainable development, Rockström et al. (2009) also proposed a framework based on boundaries that define the safe operating space (SOS) for humanity, associated with the planet's biophysical subsystems or processes. Similarly the concept of Safe Minimum Standards (SMS) as in Margolis and Naevdal (2008) relates to tipping thresholds and risky areas.

If the constraints induced by these indicators, standards, thresholds, boundaries or tipping points have to be satisfied over time with an intergenerational equity perspective, such sustainability problems can be framed into the mathematical framework of viable control (Aubin, 1990; De Lara and Doyen, 2008). Interpreting viability constraints as minimal rights to be guaranteed to all generations, the viability approach can be used to address the sustainability issue (Baumgärtner and Quaas, 2009; Doyen and Martinet, 2012) and particularly strong sustainability. This approach has been notably applied to the sustainable management of renewable resources (Béné et al., 2001; Krawczyk et al., 2013; Péreau et al., 2012; Cissé et al., 2013; Doyen et al., 2017), as recently reviewed in Schuhbauer and Sumaila (2016); Oubraham and Zaccour (2018).

Maximin and viability approaches are strongly tied since the maximin value function turns out to correspond to a 'maximal viability' (Doyen and Martinet, 2012; Martinet and Doyen, 2007). More specifically, Doyen and Martinet (2012) proved that the value function of the maximin problem is the solution of a static optimization problem under constraints involving the viability kernel. Thus, maximin trajectories are particular and extreme viable trajectories and thus inherit viability properties. However, so far, such results linking maximin and viability have been limited to a single utility or payoff in line with the usual single-valued maximin approach.

The present paper expands these ideas and findings within a multi-criteria framework using Pareto optimality. The major contribution of the paper is twofold. First, a characterization of Pareto maximin using the viability kernel is exhibited. Such a finding makes it possible to determine ecological-economic standards underlying strong sustainability and the trade-offs and/or synergies between these sustainability standards. The second major contribution of the paper is to provide algorithms and numerical schemes for multi-criteria maximin and sustainable standards based on the specific set-valued formulation of dynamic programming for viability. Two examples relying on renewable resource management and connected to well-known references points in the field, namely, MSY (maximum sustainable yield) and MEY (maximum economic yield) (Clark, 1990), illustrate the theoretical results.

The paper is organized as follows: In the next section, we formulate both maximin multi-criteria, viable control and inverse viability problems. In particular, the set of sustainable standards is defined. Then, Section 3 provides the two main results. It first notes the links between Pareto maximin, the viability kernel, and the set of sustainable standards. Second, a method based on the dynamic programming principle for computing or approximating the set of these sustainable standards or thresholds and thus Pareto maximin is derived. In Section 4, we exemplify these general findings with two stylized models inspired by renewable resource management. Section 5 summarizes our contributions and provides some perspectives. At the end of the document, the appendix gathers the detailed mathematical proofs of propositions.

2. Multi-criteria maximin and viable control problems

2.1. A general dynamic economic model

Following Fleurbaey (2015); Doyen and Martinet (2012), consider an economy described at time t by n capital stocks $x(t) \in \mathbb{R}^n$ such as natural resources or physical capital or labor, and m decision variables $c(t) \in \mathbb{R}^m$ including resource extraction, harvesting effort or consumption.

All of the economic dynamics are represented by the following controlled discrete-time dynamic system

$$\begin{cases} x(t+1) = D(x(t), c(t)) \\ x(t_0) = x_0 \\ c(t) \in C(x(t)) \end{cases} \quad (1)$$

where function D captures stocks dynamics, including resource growth or investment while x_0 stands for the initial condition of the economy and $C \subset \mathbb{R}^m$ stands for the set of admissible decisions which can potentially depend on the state. The time period is $\mathcal{T} = \{t_0, t_0+1, \ldots, T\}$ where temporal horizon $T \in \mathbb{N} \cup \{+\infty\}$ can include finite or infinite cases.

2.2. From maximin to Pareto maximin

Consider at each period t, different payoffs, metrics or scores $I_j(x(t), c(t))$, which may depend on states and controls. In economic terms, these criteria can capture utilities from the demand side or profits from the supply side.

The usual maximin approach (Solow, 1974; Cairns and Long, 2006; Doyen and Martinet, 2012) aims at maximizing the minimal level over time of a specific payoff, say I_j. In other words, the maximin criterion defines the maximal level of payoff that can be sustained given an initial endowment x_0. Hence, the maximin value function $V_j : \mathbb{R}^n \to \mathbb{R}$ associated with metrics I_j is defined by

$$V_j(x_0) \overset{\text{def}}{=} \sup_{\left\{\begin{array}{l}(x(\cdot), c(\cdot)) \\ \text{satisfying (1)}\end{array}\right.} \inf_{t \in \mathcal{T}} I_j(x(t), c(t)).$$

Whenever the supremum is reached and corresponds to a maximum,[1] this value function defines an optimal economic trajectory $x^\star(\cdot)$ together with optimal economic decisions $c^\star(\cdot)$. Now, adopting a strong sustainability viewpoint, taking into account and balancing various payoffs related to ecological and economic objectives within an inter-generational perspective, the investigation of a multi-objective maximin problem can bring major insights. It relates to a maximin optimization problem that involves multiple metrics. In mathematical terms, such a problem can be formulated as

$$\sup_{\left\{\begin{array}{l}(x(\cdot), c(\cdot)) \\ \text{satisfying (1)}\end{array}\right.} \left(\inf_{t \in \mathcal{T}} I_1(x(t), c(t)), \ldots, \inf_{t \in \mathcal{T}} I_p(x(t), c(t)) \right) \tag{2}$$

where the integer $p \geq 2$ is the number of objectives. At this stage, we have to clarify the mathematical definition of this multi-objective maximin problem (2) because generally, there does not exist a feasible trajectory that simultaneously maximizes all intertemporal criteria $\inf_{t \in \mathcal{T}} I_j(x(t), c(t))$. Here, we focus on Pareto optimal solutions, namely, solutions that cannot be improved in any of the objectives without degrading at least one of the other payoffs. In mathematical

[1] By convention, we set $V(x_0) = -\infty$ whenever the problem has no solution in the sense that, for any admissible control path $c(\cdot)$, the minimal payoff is not bounded from below, namely, $\inf_{t \in \mathcal{T}} I_j(x(t), c(t)) = -\infty$. Note also that our formulation of the maximin problem as a 'sup inf' problem is more general than the classical 'max min' formulation. In particular, it allows us to account for the cases when the supremum is not reached.

terms, a feasible solution $(x^*(\cdot), c^*(\cdot))$ satisfying (1) will thus be said to weakly dominate in the *Pareto maximin* sense another feasible solution $(x(\cdot), c(\cdot))$, if

$$\begin{cases} \inf_{t\in\mathcal{T}} I_j(x^*(t), c^*(t)) \geq \inf_{t\in\mathcal{T}} I_j(x(t), c(t)), & \forall\ j \in \{1, 2, \dots, p\} \\ \exists\ j^* \in \{1, 2, \dots, p\} \text{ s. t. } \inf_{t\in\mathcal{T}} I_{j^*}(x^*(t), c^*(t)) > \inf_{t\in\mathcal{T}} I_{j^*}(x(t), c(t)). & \end{cases}$$

A solution $(x^*(\cdot), c^*(\cdot))$ will be called a strong Pareto maximin solution of problem (2), if there is no other solution that weakly dominates it. The set of strong Pareto maximin solutions will be denoted by $\mathcal{A}_m$

$$\mathcal{A}_m(x_0, \mathcal{T}) \stackrel{\text{def}}{=} \{(x^*(\cdot), c^*(\cdot)) \text{ is a strong Pareto maximin solution}\}.$$

The set of values in terms of intertemporal criteria of these strong Pareto maximin solutions, denoted by $\mathcal{V}_m$, corresponds to

$$\mathcal{V}_m(x_0, \mathcal{T}) \stackrel{\text{def}}{=} \left\{ \left(\inf_{t\in\mathcal{T}} I_1(x^*(t), c^*(t)), \dots, \inf_{t\in\mathcal{T}} I_p(x^*(t), c^*(t)) \right) \text{ such that } (x^*(\cdot), c^*(\cdot)) \in \mathcal{A}_m(x_0, \mathcal{T}) \right\}. \tag{3}$$

Similarly to previous concepts, we say that a feasible solution $(x^*(\cdot), c^*(\cdot))$ satisfying (1) strongly dominates in the *Pareto maximin* sense another feasible solution $(x(\cdot), c(\cdot))$, if

$$\inf_{t\in\mathcal{T}} I_j(x^*(t), c^*(t)) > \inf_{t\in\mathcal{T}} I_j(x(t), c(t)) \quad \forall\ j \in \{1, 2, \dots, p\}. \tag{4}$$

A solution $(x^*(\cdot), c^*(\cdot))$ will be called a weak Pareto maximin solution, if there is no other solution that strongly dominates it in the sense of (4). Thus, the set of weak Pareto maximin solutions will be

$$\mathcal{A}_m^w(x_0, \mathcal{T}) \stackrel{\text{def}}{=} \{(x^*(\cdot), c^*(\cdot)) \text{ is a weak Pareto maximin solution}\},$$

and the set of values in terms of payoffs of these weak Pareto maximin solutions, denoted by $\mathcal{V}_m^w$, corresponds to

$$\mathcal{V}_m^w(x_0, \mathcal{T}) \stackrel{\text{def}}{=} \left\{ \left(\inf_{t\in\mathcal{T}} I_1(x^*(t), c^*(t)), \dots, \inf_{t\in\mathcal{T}} I_p(x^*(t), c^*(t)) \right) \text{ such that } (x^*(\cdot), c^*(\cdot)) \in \mathcal{A}_m^w(x_0, \mathcal{T}) \right\}. \tag{5}$$

From the previous definitions, the following inclusions can be easily derived

$$\mathcal{A}_m(x_0, \mathcal{T}) \subset \mathcal{A}_m^w(x_0, \mathcal{T}); \qquad \mathcal{V}_m(x_0, \mathcal{T}) \subset \mathcal{V}_m^w(x_0, \mathcal{T}).$$

These last assertions emphasize that strong Pareto maximin optimality is more demanding than weak Pareto maximin optimality.

2.3. The viability kernel and the set of sustainable standards

Let us now move towards the viable control approach. In such a framework, state and decision variables have to comply with inequalities involving the different payoffs, metrics or scores I_j introduced previously (e.g., utilities, profits, production, consumption, etc.)

$$I_j(x(t), c(t)) \geq I_j^{\lim} \quad \text{for} \quad j = 1, \ldots, p \tag{6}$$

where $I_j^{\lim}$ are thresholds and standards not to exceed in order to avoid crisis and to put the safety of the system at stake.

In the context introduced above, the viable control approach has traditionally been focused on analyzing and computing the viability kernel (Aubin, 1990), a key mathematical concept, which consists of the set of the initial conditions for which the system (1) is feasible with respect to constraints (6). In more mathematical terms, for a given vector of thresholds $I^{\lim} = (I_1^{\lim}, \ldots, I_p^{\lim}) \in \mathbb{R}^p$ and a given time period $\mathcal{T} = \{t_0, t_0 + 1, \ldots, T\}$, characterized by t_0 and T, the viability kernel reads as follows

$$\mathrm{Viab}^T(t_0, I^{\lim}) \stackrel{\text{def}}{=} \left\{ x_0 \;\middle|\; \begin{array}{l} \exists\ (c(t_0), \ldots, c(T)) \text{ and } (x(t_0), \ldots, x(T)) \\ \text{satisfying dynamics (1) and constraints (6)} \\ \text{ for each time } t \in \mathcal{T} \end{array} \right\} . \tag{7}$$

The viability kernel $\mathrm{Viab}^T(t_0, I^{\lim})$ associated with thresholds $I^{\lim}$, is known to play a basic role (in both continuous and discrete time) in the analysis of such problems and in the design of viable control feedbacks. Many viability works have been devoted to the characterization or computation of the viability kernel (De Lara and Doyen, 2008; Béné et al., 2001; Bonneuil and Boucekkine, 2014; Krawczyk et al., 2013; Péreau et al., 2012; Schuhbauer and Sumaila, 2016; Oubraham and Zaccour, 2018), in particular using methods based on the dynamic programming principle (De Lara et al., 2007; Doyen et al., 2013).

Less attention, with the exception of Doyen and Martinet (2012); Martinet and Doyen (2007); Martinet (2011), has been paid to the inverse viability problem consisting in determining, given an initial condition, the constraints and standards, captured by the vector of thresholds $I^{\lim}$, for which the dynamic system (1) mixed with constraints (6) is feasible. In more mathematical terms, the question consists in determining the following set:

$$\mathcal{S}^T(t_0, x_0) \stackrel{\text{def}}{=} \left\{ I^{\lim} = (I_1^{\lim}, \ldots, I_p^{\lim}) \in \mathbb{R}^p \;\middle|\; \begin{array}{l} \exists\ (c(t_0), \ldots, c(T)) \text{ and} \\ (x(t_0), \ldots, x(T)) \\ \text{satisfying dynamics (1)} \\ \text{ and constraints (6)} \end{array} \right\} . \tag{8}$$

Hereafter in this paper, this set $\mathcal{S}^T(t_0, x_0)$ is named *the set of sustainable standards* starting from x_0 at time t_0. The set $\mathcal{S}^T(t_0, x_0)$ gives major insights into the sustainability of the current state of the economy x_0 in the sense that it informs on levels of the different metrics I_j that can be guaranteed both today and in the future. In particular, a small set $\mathcal{S}^T(t_0, x_0)$ means that the room for manoeuvre in terms of sustainability is reduced and that the current state x_0 is vulnerable in that sense. The most extreme and unfavorable case occurs when the set of sustainable standards is empty[2] (e.g., $\mathcal{S}^T(t_0, x_0) = \emptyset$), which indicates that the current economic state does not make it possible the compliance of the different indicators at play.

Mathematically speaking, the above set can be considered as the inverse mapping of the viability kernel, in the sense that

$$I^{\lim} \in \mathcal{S}^T(t_0, x_0) \;\Leftrightarrow\; x_0 \in \mathrm{Viab}^T(t_0, I^{\lim}). \tag{9}$$

As for the viability kernel, identifying and computing $\mathcal{S}^T(t_0, x_0)$ is challenging. However, for applications, it may be easier to determine $\mathcal{S}^T(t_0, x_0)$ than the whole viability kernel because the current or initial state x_0 is usually known or at least estimated in the real world.

The following monotonicity properties of the set $\mathcal{S}^T(t_0, x_0)$ are useful for the identification and approximation of the set of sustainable standards:

$$\mathcal{S}^T(t, x_0) \quad \subset \quad \mathcal{S}^T(t+1, x_0) \tag{10}$$

$$\mathcal{S}^{T+1}(t, x_0) \quad \subset \quad \mathcal{S}^T(t, x_0). \tag{11}$$

Previous inclusions, whose proofs stem from the very definition of the set of sustainable standards $\mathcal{S}^T(t, x_0)$ in (8), simply stress that standards that are sustainable during a period of time, are also sustainable over a shorter period. In line with such a monotonicity property, one can deduce that the set of sustainable thresholds in the infinite horizon case can be characterized through sustainable standards in finite time as follows:

$$\mathcal{S}^\infty(t_0, x_0) = \bigcap_{T \geq t_0} \mathcal{S}^T(t_0, x_0).$$

In other words, the infinite case is just a simple (monotonic) extension of the finite cases[3]. The numerical examples investigated later in sections 4.1 and 4.2 are illustrative in that regard.

3. Main results

3.1. Weak Pareto Maximin as the optimization of viability standards

In this section, we characterize the Pareto maximin $\mathcal{V}_m(x_0, \mathcal{T})$ and the weak Pareto maximin $\mathcal{V}_m^w(x_0, \mathcal{T})$, defined by (3) and (5) respectively, through static

[2]Note that if indicators I_j are bounded from below, for instance $I_j(\cdot, \cdot) \geq 0$, then $\mathcal{S}^T(t_0, x_0)$ is never empty, because $0 \in \mathcal{S}^T(t_0, x_0)$. Nevertheless, in this example, the case $\mathcal{S}^T(t_0, x_0) = \{0\}$ represents the worst situation (i.e., almost equivalent to be empty).

[3]In Martinet et al. (2011) the authors define the set $\mathcal{S}^\infty(t_0, x_0)$, motivated by a bargaining problem with intertemporal maximin payoffs. The definition of this set, in the continuous-time framework, can be found in Martinet (2011).

multi-criteria optimization problems involving the viability kernel defined by (7) as constraint. Such a finding allows us to interpret multi-criteria maximin as an extreme case of viability.

We start from the following simple proposition.

Proposition 3.1. *Assume the existence of a weak Pareto maximin optimal solution starting from state x_0, namely, that $\mathcal{A}_m^w(x_0, \mathcal{T}) \neq \emptyset$. Then*

$$x_0 \in \bigcap_{I^{\lim} \in \mathcal{V}_m^w(x_0, \mathcal{T})} Viab^T(t_0, I^{\lim}),$$

or equivalently

$$\mathcal{V}_m^w(x_0, \mathcal{T}) \subset \mathcal{S}^T(t_0, x_0).$$

The proof of the proposition is presented in the appendix. The interpretation of this simple result is that an economic endowment x_0 makes it possible to guarantee standards belonging to the set $\mathcal{V}_m^w(x_0, \mathcal{T})$ of multi-criteria maximin. In other words, multi-criteria maximin values $\mathcal{V}_m^w(x_0, \mathcal{T})$ are viable thresholds from x_0.

We now present an important property that will be the cornerstone of our results. This proposition states that the weak Pareto maximin values of a state x_0 is the highest level (in the weak Pareto sense) of the viability constraints such that the current state lies within the underlying viability kernel. The result also holds true in the (strong) Pareto sense. Let us recall that a given vector $I = (I_1, \ldots, I_p)$ is said to be strongly Pareto dominated by $\tilde{I} = (\tilde{I}_1, \ldots, \tilde{I}_p)$ if $I < \tilde{I}$, and I is said to be (weakly) Pareto dominated by $\tilde{I}$ if $I \leq \tilde{I}$ and there exists $j^* \in \{1, \ldots, p\}$ such that $I_{j^*} < \tilde{I}_{j^*}$. With these concepts, one can define the weak Pareto boundary and the Pareto boundary of any set $\mathcal{S} \subset \mathbb{R}^p$ by

- Weak Pareto boundary of $\mathcal{S}$:

$$\mathcal{P}_w(\mathcal{S}) \stackrel{\text{def}}{=} \{I \in \mathcal{S} \mid \nexists\, \tilde{I} \in \mathcal{S} \ \text{ such that } \ \tilde{I} \ \text{ strongly dominates } \ I\};$$

- (Strong) Pareto boundary of $\mathcal{S}$:

$$\mathcal{P}(\mathcal{S}) \stackrel{\text{def}}{=} \{I \in \mathcal{S} \mid \nexists\, \tilde{I} \in \mathcal{S} \ \text{ such that } \ \tilde{I} \ \text{ dominates } \ I\}.$$

We obtain the following characterizations of the Pareto maximin $\mathcal{V}_m^w$ with respect to the Pareto boundary of $\mathcal{S}$.

Proposition 3.2. *For any initial conditions x_0, we have*

$$\begin{aligned} \mathcal{V}_m^w(x_0, \mathcal{T}) &= \mathcal{P}_w\Big(\{I^{\lim} \mid x_0 \in Viab^T(t_0, I^{\lim})\}\Big) = \mathcal{P}_w\Big(\mathcal{S}^T(t_0, x_0)\Big) \\ \mathcal{V}_m(x_0, \mathcal{T}) &= \mathcal{P}\Big(\{I^{\lim} \mid x_0 \in Viab^T(t_0, I^{\lim})\}\Big) = \mathcal{P}\Big(\mathcal{S}^T(t_0, x_0)\Big). \end{aligned}$$

The second equality of Proposition 3.2 was already stated in the preprint Martinet et al. (2011) in the context of infinite horizon and strong Pareto optimality. The generalization to weak Pareto optimality and to finite horizon case underlying the Proposition 3.2 is proved in the appendix.

Regarding the understanding of the above proposition, let us first remark that the equality

$$\{I^{\lim} \mid x_0 \in \mathrm{Viab}^T(t_0, I^{\lim})\} = \mathcal{S}^T(t_0, x_0)$$

directly arises from (9). The result of Proposition 3.2 can be interpreted as follows. We know that a utility level $I^{\lim}$ is sustainable from initial state x_0 if x_0 belongs to the viability kernel $\mathrm{Viab}^T(t_0, I^{\lim})$. The higher are the level of scores $I^{\lim}$ to guarantee, the smaller is the viability kernel $\mathrm{Viab}^T(t_0, I^{\lim})$ and thus the less numerous are the sustainable initial economic states. The maximal sustainable standards (multi-criteria maximin values) correspond to the highest guaranteed payoffs for which the related viability kernel contains the initial state x_0. Said differently, the different sustained payoffs level $I_j^{\lim}$ are increased until the state x_0 lies on the boundary of the viability kernel. In contrast, when this is not possible, the state x_0 is located in the interior of the viability kernel.[4] Proposition 3.2 also obviously means that, from a given x_0, no payoffs greater than weak Pareto maximin values $\mathcal{V}_m^w(x_0, \mathcal{T})$ or Pareto maximin values $\mathcal{V}_m(x_0, \mathcal{T})$ can be guaranteed.

Although the interpretation of this proposition is rather simple, it has major consequences. It indicates that the multi-criteria maximin values can be defined within the viability framework using a multi-criteria optimization problem on the viability kernel. Even if this method is not necessarily simpler than the standard approach to solve maximin problems, it allows for exhibiting some hidden properties of the maximin solutions. Whenever the solution of a given optimization problem can be formulated in terms of a viability kernel, the solution indeed inherits the properties of the kernel. Such properties include a dynamic programming structure that we use to derive numerical approximations of both the multi-criteria maximin and sustainable standards.

3.2. Dynamic programming property

As stated previously in Proposition 3.2, the computation of sustainable standards or thresholds $\mathcal{S}^T(t_0, x_0)$ is crucial for the identification of Pareto maximin values $\mathcal{V}_m^w(x_0, \mathcal{T})$ and $\mathcal{V}_m(x_0, \mathcal{T})$. To identify or approximate this set, it turns out that we can rely on a dynamic programming (DP) structure. Such a DP structure stems from the DP underlying the viability kernel in discrete-time as detailed in De Lara and Doyen (2008)

$$\mathrm{Viab}^T(t) = \{x,\ \exists\, c \in C(x) \text{ s. t. } D(x, c) \in \mathrm{Viab}^T(t+1)\}. \tag{12}$$

[4] In such cases, the maximin path is said to be non-regular (Cairns and Long, 2006).

The essence of DP is thus to propose an iterative method starting from the temporal horizon T and then moving backward with respect to time. Combining this DP structure of the viability kernel with the inverse relationship (9) between the viability kernel and sustainable thresholds $\mathcal{S}^T(t,x)$, we deduce a DP structure for the sustainability standards in $\mathcal{S}^T(t_0,x_0)$. Focusing on the finite horizon case, it reads as follows:

Proposition 3.3. *Assume that $T < +\infty$. For every $x \in \mathbb{R}^n$ one has*

$$\mathcal{S}^T(T,x) = \bigcup_{c \in C(x)} \{I^{\lim} \in \mathbb{R}^p \mid I(x,c) \geq I^{\lim}\} \tag{13}$$

and for every $t = t_0, \dots, T-1$

$$\mathcal{S}^T(t,x) = \bigcup_{c \in C(x)} \mathcal{S}^T(t+1, D(x,c)) \cap \{I^{\lim} \in \mathbb{R}^p \mid I(x,c) \geq I^{\lim}\}. \tag{14}$$

The proof of this proposition is detailed in the appendix. As for usual optimal control problem, the interest of DP is to offer a decomposition of a problem over T periods into T more simple problems. Another well-known interest of DP is to provide feedback and closed-loop controls as compared to open-loop controls. Such feedback controls constitute a major ingredient of adaptive management. From the geometrical result displayed in Proposition 3.3, an equivalent functional characterization can be derived. For this purpose, we introduce the indicator function associated to any set $\mathcal{S} \subset \mathbb{R}^p$, that is

$$\mathbb{1}_{\mathcal{S}}(I^{\lim}) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if} \quad I^{\lim} \in \mathcal{S} \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Then, we define the function $V(t,x,\cdot) : \mathbb{R}^p \longrightarrow \{0,1\}$ for any state $x \in \mathbb{R}^n$ and for every time $t = t_0, \dots, T$ by

$$V(t,x,I^{\lim}) \stackrel{\text{def}}{=} \mathbb{1}_{\mathcal{S}^T(t,x)}(I^{\lim}) \qquad \text{for all} \quad I^{\lim} \in \mathbb{R}^p. \tag{16}$$

Using these Boolean functions, the dynamic programming characterization of sustainability standards described in the previous Proposition 3.3 now reads as follows.

Proposition 3.4. *Assume that $T < +\infty$ and $C(x) \neq \emptyset$ for every $x \in \mathbb{R}^n$. Then the function $V(t,x,\cdot)$ defined in (16) is the solution of the following backward dynamic programming equation:*

$$\begin{cases} V(T,x,I^{\lim}) = \sup\limits_{c \in C(x)} \mathbb{1}_{\mathbb{R}^p_+}(I(x,c) - I^{\lim}) \\ V(t,x,I^{\lim}) = \sup\limits_{c \in C(x)} \mathbb{1}_{\mathbb{R}^p_+}(I(x,c) - I^{\lim}) V(t+1, D(x,c), I^{\lim}) \quad \forall \;\; t_0 \leq t < T. \end{cases} \tag{17}$$

The proof of this proposition is presented in the appendix. The interest of the functional formulation in (17) is twofold. First, it highlights to what extent the problem of determining sustainable standards corresponds to a specific and non-smooth optimization problem involving Boolean functions. Second, (17) provides a rather simple method to implement numerically the computation of sustainable standards in $\mathcal{S}^T(t, x_0)$ and thus of multi-criteria maximin $\mathcal{V}_m(x_0, \mathcal{T})$ because it is easier on computers to address functions instead of sets. Nevertheless, this method, as all involving DP, is known to suffer from the curse of dimensionality.

4. Examples

In this section, we illustrate both the interest and the computation of the set of sustainable standards $\mathcal{S}^T(t_0, x)$ for two examples based on renewable resources management inspired by Clark (1990) and their viability counterpart, as in Béné et al. (2001); Krawczyk et al. (2013); Péreau et al. (2012); Cissé et al. (2013); Schuhbauer and Sumaila (2016); Doyen et al. (2017). In Example 4.1, a regulating agency aims at the conservation and sustainable harvesting of the renewable resource while in Example 4.2 both profitability and stock conservation are taken into account. First case relates to Maximum Sustainable Yield (MSY) strategy intensively used for fisheries management such as the PCF (European Common Policy for Fisheries). The second example is connected to Maximum Economic Yield (MEY), a reference point that also plays a major role in fisheries regulation for several countries worldwide including Australia.

For both examples, the stock of the renewable resource at time t is represented by $x(t) \geq 0$, and its dynamics with harvesting $h(t)$ is described by

$$x(t+1) = f(x(t)) - h(t),$$

where f stands for the renewable function of the stock. Usual examples of such population dynamics f include logistic, Beverton-Holt, Gomperz or Ricker growth.

4.1. Sustainable standards for stock and harvesting

Consider first that the regulating agency intends to compute the multi-criteria maximin in terms of both stock and harvesting:

$$\mathcal{V}_m(x_0, \mathcal{T}) = \sup_{\left\{\begin{array}{l}(x(\cdot), h(\cdot)) \text{ satisfying} \\ x(t+1) = f(x(t)) - h(t) \\ x(t_0) = x_0\end{array}\right.} \left(\inf_{t \in \mathcal{T}} x(t), \inf_{t \in \mathcal{T}} h(t)\right). \tag{18}$$

Said differently, the social objective of the resource management consists in ensuring both current stock and catch. Reformulating the problem from a viability viewpoint, maximin multicriteria problem (18) relates to sustaining both stock and catch through thresholds $x^{\lim}$ and $h^{\lim}$ as follows

$$\begin{cases} x(t+1) = f(x(t)) - h(t), \ x(t_0) = x_0 \\ x(t) \geq x^{\lim} > 0 \\ h(t) \geq h^{\lim} > 0. \end{cases} \tag{19}$$

As claimed by Proposition 3.2, the identification of viable thresholds $x^{\lim}$ and $h^{\lim}$ with respect to current state x_0 corresponds directly to the computation of the set $\mathcal{S}^T(t_0, x_0)$ and indirectly to the viability kernel $\text{Viab}^T(t_0, x^{\lim}, h^{\lim})$.

It turns out that MSY level plays a major role as a tipping point in the determination of the viability kernels and sustainable thresholds. In line with that, we denote by $\sigma(x)$ the harvest level at equilibrium given a biomass level x, that is

$$\sigma(x) = f(x) - x$$

and MSY, the catch level at equilibrium where this quantity is maximized,

$$\text{MSY} \stackrel{\text{def}}{=} \max_{\text{x} \in [0,\text{K}]} \sigma(\text{x}).$$

To compute analytically $\mathcal{S}^T(t_0, x_0)$ and the viability kernel $\text{Viab}^T(t_0, x^{\lim}, h^{\lim})$, it is convenient to consider the Beverton-Holt population dynamics, namely,

$$f(x) = (1+r)x \left(1 + \frac{r}{K}x\right)^{-1} \tag{20}$$

where the intrinsic growth r and carrying capacity K are two positive parameters. For this Beverton-Holt growth function (20), the MSY biomass level x_{MSY} at equilibrium is given by

$$x_{\text{MSY}} = \frac{K}{1 + \sqrt{1+r}}.$$

The viability kernel $\text{Viab}^\infty(t_0, x^{\lim}, h^{\lim})$ is identified in De Lara and Doyen (2008). Using (9), we then derive analytically the set $\mathcal{S}^\infty(t_0, x_0)$, which is given by

$$\mathcal{S}^\infty(t_0, x_0) = \{(x^{\lim}, h^{\lim}) \mid 0 \leq x^{\lim} \leq \min\{x_0, K\}; \quad 0 \leq h^{\lim} \leq \sigma(x^{\lim})\}.$$

Figure 1 displays both, the set of sustainable thresholds $\mathcal{S}^\infty(t_0, x_0)$ and its numerical approximation for different initial conditions in the following cases: (i) $0 < x_0 < x_{\text{MSY}}$; (ii) $x_{\text{MSY}} < x_0 < K$; and (iii) $K < x_0$. First row plots the analytic solution of $\mathcal{S}^\infty(t_0, x_0)$. The second, third, and fourth rows show the numerical approximation of $\mathcal{S}^T(t, x_0)$ for $t = T$ (second row), $t = T/2$ (third row), and $t = t_0$ (fourth row), where $t_0 = 0$ and $T = 20$. This procedure was conducted using the backward dynamic programming equation established in Proposition 3.4. For this numerical example, we use the values $r = 1.75$ and $K = 50$ for the parameters underlying population renewal f.

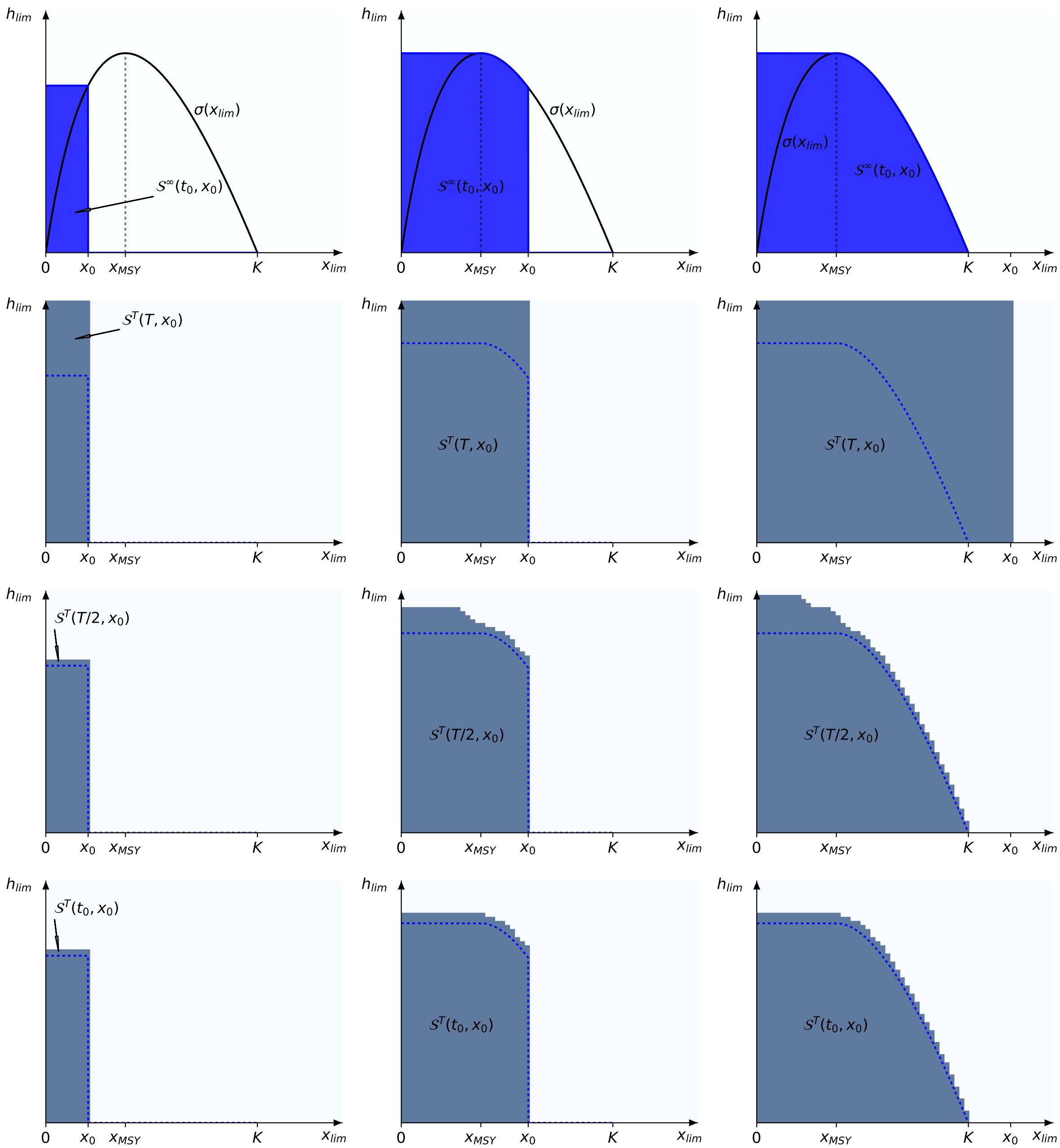

Figure 1: Sustainable thresholds $\mathcal{S}^\infty(t_0, x_0)$ (first row) and their approximations (second, third and fourth rows) $\mathcal{S}^{20}(t_0, x_0)$ for different initial conditions x_0: first column: $x_0 < x_{\mathrm{MSY}}$; second column: $x_{\mathrm{MSY}} < x_0 < K$; third column: $x_0 > K$. The numerical approximations based on dynamic programming as in Proposition 3.4 relate on different initial times t_0: second row: $t_0 = T = 20$; third row: $t_0 = T/2 = 10$, fourth row: $t_0 = 0$. The blue dashed lines stand for the exact boundaries of sustainable thresholds $\mathcal{S}^\infty(t_0, x_0)$. Parameters of stock dynamics are set to $r = 1.75$ and $K = 50$.

Using Proposition 3.2 along with the previous computation of $\mathcal{S}^\infty(t_0, x_0)$, we deduce the (strong) Pareto maximin boundary

$$\mathcal{V}_m(x_0, \mathcal{T}) = \mathcal{P}\Big(\mathcal{S}^\infty(t_0, x_0)\Big) = \begin{cases} (x_0, \sigma(x_0)) & \text{if} \quad x_0 \leq x_{\text{MSY}} \\ \{(x, \sigma(x)) \mid \ x_{\text{MSY}} \leq x \leq x_0\} & \text{if} \quad x_{\text{MSY}} \leq x_0 \leq K \\ \{(x, \sigma(x)) \mid \ x_{\text{MSY}} \leq x \leq K\} & \text{if} \quad K \leq x_0 \end{cases}$$

where $\mathcal{T} = \{t_0, t_0 + 1, \ldots\}$.

Interestingly, two contrasted situations for strong maximin multicriteria can be distinguished with respect to the so-called maximum sustainable stock x_{MSY}. In the first case (first column in the Figure 1) where the stock is in a situation of biological overexploitation in the sense that $x_0 < x_{\text{MSY}}$, there is a synergy between conservation and harvesting because the strong Pareto boundary of $\mathcal{S}^\infty(t_0, x_0)$, located on the right-upper corner of the set, is reduced to a unique point $(x_0, \sigma(x_0))$.

In contrast, when the stock is biologically underharvested $x_0 > x_{\text{MSY}}$ as in the second or third cases (second and third columns of Figure 1), a trade-off emerges between biological and production sustainability because the strong Pareto boundary of $\mathcal{S}^\infty(t_0, x_0)$ corresponds to the decreasing concave curve σ located on the right-upper part of the set. In other words, rising stock conservation requirements x^{lim} alters the sustainable productive scores h^{lim} and conversely.

4.2. Sustainable standards for stock and profit

In this second example, the decision variable is the harvesting effort (for instance number of vessels in a fishery) denoted by $e(t)$, and the regulating agency aims at securing levels of biomass and profit. In the vein of the Schaefer model intensively used in the literature of renewable resources, catches are assumed to be linear with respect to both effort and stock, namely,

$$h(t) = qe(t)x(t)$$

where $q > 0$ is the so-called catchability parameter. Profit $\pi(t)$ are defined as the difference between incomes induced by harvesting and cost of effort with

$$\pi(t) = pqe(t)x(t) - ce(t)$$

where $p > 0$ is the price per unit of biomass, and $c > 0$ is the cost per unit of the harvesting effort. Thus, the sustainability problem of the regulating agency reads as follows

$$\begin{cases} x(t+1) = f(x(t)) - qe(t)x(t) \\ x(t_0) = x_0 \\ e(t) \geq 0 \\ x(t) \geq x^{\text{lim}} \geq 0 \\ pqe(t)x(t) - ce(t) \geq \pi^{\text{lim}} \geq 0 \end{cases} \tag{21}$$

Reformulating the problem in terms of sustainable standards and multicriteria maximin, the objective is, given an initial biomass level $x_0 \geq 0$, to compute the set $\mathcal{S}^T(t_0, x_0)$ containing thresholds $x^{\lim} \geq 0$ and $\pi^{\lim} \geq 0$ for which there are states and control trajectories $(x(\cdot), e(\cdot))$ satisfying dynamics and constraints (21).

In this example, we consider the logistic growth for the renewable resource given by

$$f(x) = x + rx\left(1 - \frac{x}{K}\right), \tag{22}$$

where r and K are again two positive parameters. As in the previous example, we denote by $\sigma(x)$ the harvest at equilibrium for a biomass level x, that is $\sigma(x) = f(x) - x$, and x_{MSY} the resource level where this quantity is maximized, which for the logistic function (22) is

$$x_{\text{MSY}} = \max_{x \in [0,K]} \sigma(x) = \frac{K}{2}.$$

Now let us denote by $\widehat{\sigma}(x)$ the profit obtained in an equilibrium stock level $x > 0$, that is

$$\widehat{\sigma}(x) \stackrel{\text{def}}{=} \sigma(x)\left(p - \frac{c}{qx}\right).$$

It turns out that the stock x_{MEY} (*Maximum Economic Yield*) maximizing the profit at equilibrium $\widehat{\sigma}(x)$ is given by

$$x_{\text{MEY}} = \frac{K}{2} + \frac{c}{2pq} > x_{\text{MSY}}.$$

In Figure 2, we plot the set $\mathcal{S}^T(t_0, x_0)$ for $T = t_0 + 20$ and initial conditions in the following cases: (i) $0 < x_0 < x_{\text{MSY}}$; (ii) $x_{\text{MEY}} < x_0 < K$; and (iii) $K < x_0$. Computations are conducted using the backward dynamic programming equation established in Proposition 3.4 and specific numerical values for parameters $r = 1.75$, $K = 50$, $q = 0.4$, $p = 0.5$, $c = 1.5$.

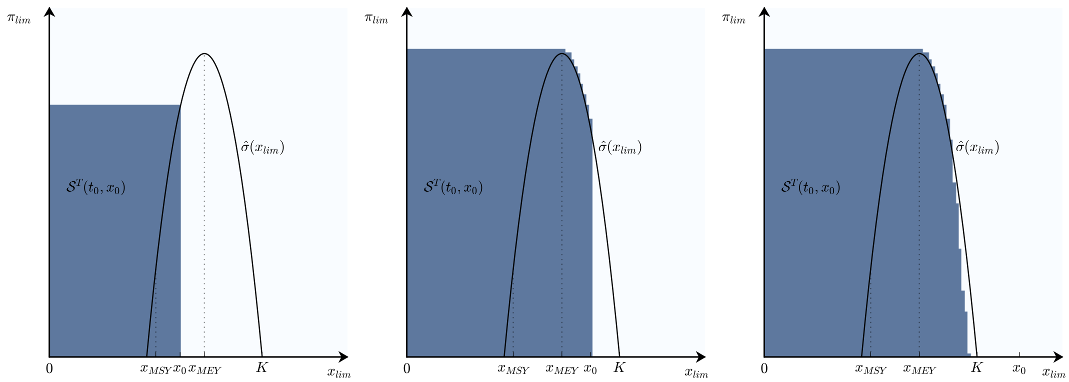

Figure 2: Sustainable thresholds $\mathcal{S}^T(t_0, x_0)$ for different initial conditions x_0: first column: $x_0 < x_{\rm MEY}$; second column: $x_{\rm MEY} < x_0 < K$; third column: $x_0 > K$; with initial time $t_0 = 0$ and horizon $T = 20$. Numerical values for the profit given in (21) and the renewal process f provided in (22) are $r = 1.75$, $K = 50$, $q = 0.4$, $p = 0.5$, $c = 1.5$.

In the light of numerical results plotted in Figure 2, we conjecture that the analytical formulation of the set of sustainable standards $\mathcal{S}^\infty(t_0, x_0)$ is given by

$$\mathcal{S}^\infty(t_0, x_0) = \{(x^{\rm lim}, \pi^{\rm lim}) \mid 0 \le x^{\rm lim} \le \min\{x_0, K\}; \quad 0 \le \pi^{\rm lim} \le \widehat{\sigma}(x^{\rm lim})\},$$

which suggests a strong qualitative analogy with the previous example presented in Section 4.1. Similarly, we conjecture that the Pareto maximin is defined by

$$\mathcal{V}_m(x_0, \mathcal{T}) = \mathcal{P}\Big(\mathcal{S}^\infty(t_0, x_0)\Big) = \begin{cases} (x_0, \widehat{\sigma}(x_0)) & \text{if} \quad x_0 \le x_{\rm MEY} \\ \{(x, \widehat{\sigma}(x)) \mid \; x_{\rm MEY} \le x \le x_0\} & \text{if} \quad x_{\rm MEY} \le x_0 \le K \\ \{(x, \widehat{\sigma}(x)) \mid \; x_{\rm MEY} \le x \le K\} & \text{if} \quad K \le x_0 \end{cases}$$

where $\mathcal{T} = \{t_0, t_0 + 1, \ldots\}$.

Hence, two contrasted situations for strong maximin multicriteria can again be distinguished but in this case with respect to the so-called maximum economic stock $x_{\rm MEY}$ instead of $x_{\rm MSY}$. Said differently, in the case of economic overexploitation characterized by $x < x_{\rm MEY}$, there is a synergy between conservation and profit because the strong Pareto boundary $\mathcal{V}_m(x_0, \mathcal{T})$ of sustainable standards $\mathcal{S}^\infty(t_0, x_0)$ coincides with $(x_0, \widehat{\sigma}(x_0))$ while, in contrast, when the stock is economically underharvested in the sense that $x_0 > x_{\rm MEY}$, a trade-off emerges through decreasing curve $\widehat{\sigma}$ between stock conservation and profit sustainability.

5. Conclusion

Operationalizing sustainability is a major challenge in terms of criteria and standards. In that respect, the maximin criterion plays an important role as a maximal score that can be sustained from an intergenerational viewpoint

(Cairns and Long, 2006; Asheim, 2007). As such, it promotes intergenerational equity. To go further, an important challenge to quantify sustainability and operationalize it in real economies is to extend such a maximin approach to multi-criteria context aiming in particular at balancing economic, ecological and social objectives without substitutability between these goals. Such a non-substitutability can occur for instance for ecosystem services without clear monetary values such as biodiversity (Doyen et al., 2013).

Accordingly, in this paper, we extend the maximin approach to multi-criteria contexts combining the viable control approach and Pareto optimality. The viability approach focuses on the consistency between a dynamic system and given constraints generally relying on standards and thresholds. Such an approach makes it possible to examine all the economic trajectories sustaining different payoffs levels over time. The paper firstly exhibits and expands the strong links between *multi-criteria maximin* and the *viability* approach. In particular, we prove how the values of the multi-criteria maximin problem are solutions of a static multi-criteria optimization problem under constraints, involving the so-called viability kernel. In other words, multi-criteria maximin emerges as an Pareto optimal viability. Such a result indicates that sustainable economic, ecological or social standards and trade-offs or synergies between these standards can be identified from the viability kernel and in particular from its boundaries. Second, drawing on the dynamic programming structure underpinning the viability kernel, the present paper provides algorithms for the identification and approximation of the sustainability standards and thus the Pareto maximin. Finally, applied models for the management of renewable resources exemplify the theoretical and general findings. These examples illustrate in particular how the trade-off or synergies between standards jointly sustaining conservation, production and profitability performances strongly depends on well-known MSY or MEY reference points.

Our results confirm the relevance of the viability approach to address the sustainability issue and address both intergenerational equity and multi-criteria objectives (Fleurbaey, 2015; Doyen and Martinet, 2012; Oubraham and Zaccour, 2018). In particular, viability defines decisions that satisfy the sustainability constraints now and maintains the capability of the economy to satisfy these constraints in the future. From that point of view, the viability approach is consistent with the Brundtland definition of sustainability characterizing sustainable development as development "that meets the needs of the present without compromising the ability of future generations to meet their own needs." In our paper, these needs clearly relate to different requirements with weak substitutability and in particular to economic (food security, profitability) and ecological (biodiversity conservation) outcomes. Thus, our results bring important methodological insights into the operationalization of strong sustainability (Baumgärtner and Quaas, 2009; Neumayer, 2010).

Acknowledgments

The authors are grateful to Sebastián Torres (UTFSM, Chile) for having contributed to the implementation of dynamic programming algorithm presented in Proposition 3.4 applied to examples 4.1 and 4.2.

This research benefited from the financial support of MathAmsud 18-MATH-05, FONDECYT grant N 1160567, and the Belmont Forum through the network SEAVIEW (ANR-14-JPF1-0003). The role of the research projects ACROSS (ANR-14-CE03-0001), NAVIRE (Cluster of Excellence COTE, ANR-10-LABX-45) and OYAMAR (FEDER) was also decisive. The second author was also partially supported by the Basal Project CMM Universidad de Chile.

References

G.B. Asheim. *Justifying, Characterizing and Indicating Sustainability*. Sustainability, Economics, and Natural Resources. Springer-Verlag, Dordrecht, 2007.

J.-P. Aubin. A survey of viability theory. *SIAM J. Control Optim.*, 28(4): 749–788, 1990.

S. Baumgärtner and M. F. Quaas. Ecological-economic viability as a criterion of strong sustainability under uncertainty. *Ecological Economics*, 68(7):2008 – 2020, 2009.

C. Béné, L. Doyen, and D. Gabay. A viability analysis for a bio-economic model. *Ecological Economics*, 36(3):385 – 396, 2001.

N. Bonneuil and R. Boucekkine. Viable ramsey economies. *Canadian Journal of Economics/Revue canadienne d'économique*, 47(2):422–441, 2014.

T. Bruckner, G. Petschel-Held, F.L. Tóth, H.-M. Füssel, C. Helm, M. Leimbach, and H.-J. Schellnhuber. Climate change decision-support and the tolerable windows approach. *Environmental Modeling and Assessment*, 4(4):217–234, 1999.

R. D. Cairns and N. V. Long. Maximin: a direct approach to sustainability. *Environment and Development Economics*, 11(3):275–300, 2006.

G. Chichilnisky. An axiomatic approach to sustainable development. *Social Choice and Welfare*, 13(2):231–257, 1996.

A.A. Cissé, S. Gourguet, L. Doyen, F. Blanchard, and J.-C. Péreau. A bio-economic model for the ecosystem-based management of the coastal fishery in french guiana. *Environment and Development Economics*, 18(3):245–269, 2013.

C. W. Clark. *Mathematical bioeconomics*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1990.

M. De Lara and L. Doyen. *Sustainable management of natural resource: mathematical models and methods*. Springer, New York, 2008.

M. De Lara, L. Doyen, T. Guilbaud, and M.-J. Rochet. Monotonicity properties for the viable control of discrete-time systems. *Systems Control Lett.*, 56(4): 296–302, 2007.

L. Doyen and V. Martinet. Maximin, viability and sustainability. *J. Econom. Dynam. Control*, 36(9):1414–1430, 2012.

L. Doyen, A. Cissé, S. Gourguet, L. Mouysset, P.-Y. Hardy, C. Béné, F. Blanchard, F. Jiguet, J.-C. Pereau, and O. Thébaud. Ecological-economic modelling for the sustainable management of biodiversity. *Computational Management Science*, 10(4):353–364, 2013.

L. Doyen, C. Béné, M. Bertignac, F. Blanchard, A. A. Cissé, C. Dichmont, S. Gourguet, O. Guyader, P.-Y. Hardy, S. Jennings, L. R. Little, C. Macher, D. J. Mills, A. Noussair, S. Pascoe, J.-C. Pereau, N. Sanz, A.-M. Schwarz, T. Smith, and O Thébaud. Ecoviability for ecosystem-based fisheries management. *Fish and Fisheries*, 18(6):1056–1072, 2017.

M. Fleurbaey. On sustainability and social welfare. *Journal of Environmental Economics and Management*, 71:34 – 53, 2015.

R. Howarth. Sustainability under uncertainty: A deontological approach. *Land Economics*, 71(4):417–427, 1995.

J. B. Krawczyk, A. Pharo, O. S. Serea, and S. Sinclair. Computation of viability kernels: a case study of by-catch fisheries. *Comput. Manag. Sci.*, 10(4):365–396, 2013.

M. Margolis and E. Naevdal. Climate change decision-support and the tolerable windows approach. *Environmental Resource Economics*, 4(40):401, 2008. doi: 10.1007/s10640-007-9162-z.

V. Martinet. A characterization of sustainability with indicators. *J. Environ. Econ. Manag.*, 61:183–197, 2011.

V. Martinet and L. Doyen. Sustainability of an economy with an exhaustible resource: A viable control approach. *Resource and Energy Economics*, 29(1): 17 – 39, 2007.

V. Martinet, P. Gajardo, M. De Lara, and H. Ramırez C. Bargaining with intertemporal maximin payoffs. EconomiX Working Papers 2011-7, University of Paris West - Nanterre la Défense, EconomiX, 2011.

E. Neumayer. *Weak versus Strong Sustainability: Exploring the Limits of Two Opposing Paradigms*. Edward Elgar Publishing, third edition, 2010.

A. Oubraham and G. Zaccour. A survey of applications of viability theory to the sustainable exploitation of renewable resources. *Ecological Economics*, 145(Supplement C):346 – 367, 2018.

J.-C. Péreau, L. Doyen, L.R. Little, and O. Thébaud. The triple bottom line: Meeting ecological, economic and social goals with individual transferable quotas. *Journal of Environmental Economics and Management*, 63(3):419 – 434, 2012.

J. Rockström, W. Steffen, K. Noone, A Persson, F. S. Chapin III, E. F. Lambin, T. M. Lenton, M. Scheffer, C. Folke, H. J. Schellnhuber, B. Nykvist, Cynthia A. de Wit, T. Hughes, S. van der Leeuw, H. Rodhe, S. Sörlin, P. K. Snyder, R. Costanza, U. Svedin, M. Falkenmark, L. Karlberg, R. W. Corell, V. J. Fabry, J. Hansen, B. Walker, D. Liverman, K. Richardson, P. Crutzen, and J. A. Foley. A safe operating space for humanity. *Nature*, 461:472 – 475, 09 2009.

A. Schuhbauer and U. Sumaila. Economic viability and small-scale fisheries - a review. *Ecological Economics*, 124(C):69–75, 2016.

R. Solow. Intergenerational equity and exhaustible resources. *Review of Economic Studies*, 41(5):29–45, 1974.

Appendix

Proof of Proposition 3.1

Assume the existence of a weak Pareto maximin optimal solution $(x^\star(\cdot), c^\star(\cdot)) \in \mathcal{A}_m^w(x_0, \mathcal{T})$ with value $I^{\lim} = (V_1(x_0), \ldots, V_p(x_0)) \in \mathcal{V}_m^w(x_0, \mathcal{T})$, where

$$V_j(x_0) = \inf_{t=t_0,\ldots,T} I_j(x^\star(t), c^\star(t)) \qquad j = 1, \ldots, p.$$

Consequently for any $j = 1, \ldots, p$

$$I_j(x^\star(t), c^\star(t)) \geq V_j(x_0), \quad \forall\, t \in t = t_0, \ldots, T.$$

In other words, state x_0 belongs to the viability kernel $\mathrm{Viab}^T(t_0, I^{\lim})$.

Proof of Proposition 3.2

First, let us prove the equality

$$\mathcal{V}_m^w(x_0, \mathcal{T}) = \mathcal{P}_w\Big(\mathcal{S}^T(t_0, x_0)\Big).$$

For $I^{\lim} \in \mathcal{V}_m^w(x_0, \mathcal{T})$ take $(x^\star(\cdot), c^\star(\cdot)) \in \mathcal{A}_m^w(x_0, \mathcal{T})$ such that

$$I^{\lim} = \left(\inf_{t=t_0,\ldots,T} I_1(x^\star(t), c^\star(t)), \ldots, \inf_{t=t_0,\ldots,T} I_p(x^\star(t), c^\star(t)) \right).$$

From Proposition 3.1 one has $I^{\lim} \in \mathcal{S}^T(t_0, x_0)$. If $I^{\lim} \notin \mathcal{P}_w\Big(\mathcal{S}^T(t_0, x_0)\Big)$, then there exists $(\tilde{x}(\cdot), \tilde{c}(\cdot))$ with

$$\tilde{V}(x_0) = \Big(\inf_{t=t_0,\ldots,T} I_1(\tilde{x}(t), \tilde{c}(t)), \ldots, \inf_{t=t_0,\ldots,T} I_p(\tilde{x}(t), \tilde{c}(t))\Big)$$

such that $\tilde{V}(x_0) > I^{\lim}$. That means that $(\tilde{x}(\cdot), \tilde{c}(\cdot))$ Pareto maximin strongly dominates $(x^\star(\cdot), c^\star(\cdot))$, which is a contradiction with $(x^\star(\cdot), c^\star(\cdot)) \in \mathcal{A}_m^w(x_0, \mathcal{T})$.

Now pick up $I^{\lim} = (I_1^{\lim}, \ldots, I_p^{\lim}) \in \mathcal{P}_w\Big(\mathcal{S}^T(t_0, x_0)\Big)$ and $(x^\star(\cdot), c^\star(\cdot))$ such that

$$I_j^{\lim} \le I_j(x^\star(t), c^\star(t)) \quad t = t_0, \ldots, T; \quad j = 1, \ldots, p.$$

Define

$$V(x_0) = \Big(\inf_{t=t_0,\ldots,T} I_1(x^\star(t), c^\star(t)), \ldots, \inf_{t=t_0,\ldots,T} I_p(x^\star(t), c^\star(t))\Big) \ge I^{\lim}.$$

If $(x^\star(\cdot), c^\star(\cdot)) \notin \mathcal{A}_m^w(x_0, \mathcal{T})$, then there exists $(\tilde{x}(\cdot), \tilde{c}(\cdot))$ with

$$\tilde{V}(x_0) = \Big(\inf_{t=t_0,\ldots,T} I_1(\tilde{x}(t), \tilde{c}(t)), \ldots, \inf_{t=t_0,\ldots,T} I_p(\tilde{x}(t), \tilde{c}(t))\Big)$$

such that $\tilde{V}(x_0) > V(x_0) \ge I^{\lim}$. This is a contradiction, because $\tilde{V}(x_0) \in \mathcal{S}^T(t_0, x_0)$ and $I^{\lim} \in \mathcal{P}_w\Big(\mathcal{S}^T(t_0, x_0)\Big)$.

Proof of Proposition 3.3

It is clear that when the initial time t is equal to the final time T, then the set $\mathcal{S}^T(T, x)$ is equal to $\bigcup_{c \in C(x)} \{I^{\lim} \in \mathbb{R}^p \mid I(x, c) \ge I^{\lim}\}$, because, from the definition of $\mathcal{S}^T(t, x)$ in (8), only the constraint $I(x, c) \ge I^{\lim}$ must be satisfied for some control $c \in C(x)$ in the single period $t = T$.

For $t_0 \le t < T$, if $I^{\lim} \in \mathcal{S}^T(t, x)$, there exists $(c(t), \ldots, c(T))$ and $(x(t_0), \ldots, x(T))$ such that

$$\begin{cases} x(s+1) = D(x(s), c(s)) & s = t, t+1, \ldots, T-1 \\ x(t) = x & \\ c(s) \in C(x(s)) & s = t, t+1, \ldots, T, \\ I(x(s), c(s)) \ge I^{\lim} & s = t, t+1, \ldots, T. \end{cases}$$

The previous statement is equivalent to say that for $c = c(t) \in C(x(t)) = C(x)$ one has $I^{\lim} \in \mathcal{S}^T(t+1, D(x, c))$, and $I(x, c) = I(x(t), c(t)) \ge I^{\lim}$, proving thus the desired equality

$$\mathcal{S}^T(t, x) = \bigcup_{c \in C(x)} \mathcal{S}^T(t+1, D(x, c)) \cap \{I^{\lim} \in \mathbb{R}^p \mid I(x, c) \ge I^{\lim}\}.$$

Proof of Proposition 3.4

The first equality in (17) stems directly from the equivalences

$$V(T,x,I^{\lim}) = 1 \ \Leftrightarrow\ I^{\lim} \in \mathcal{S}^T(T,x) \ \Leftrightarrow\ \exists\, c \in C(x) \ \text{ such that } \ I(c,x) \geq I^{\lim},$$

where the last equivalence is obtained from (13).

For $t_0 \leq t < T$, first assume $I^{\lim} \in \mathcal{S}^T(t,x)$ (i.e., $V(t,x,I^{\lim}) = 1$). From Proposition 3.3 we can deduce there exists $c \in C(x)$ such that

$$I^{\lim} \in \mathcal{S}^T(t+1, D(x,c)) \cap \{I^{\lim} \in \mathbb{R}^p \mid I(x,c) \geq I^{\lim}\}$$

and therefore

$$\begin{aligned}
1 &= \mathbb{1}_{\mathbb{R}^p_+}(I(x,c) - I^{\lim})\, \mathbb{1}_{\mathcal{S}^T(t+1,x)}(I^{\lim}) \\
&= \mathbb{1}_{\mathbb{R}^p_+}(I(x,c) - I^{\lim}) V(t+1, D(x,c), I^{\lim}) \\
&\leq \sup_{c \in C(x)} \mathbb{1}_{\mathbb{R}^p_+}(I(x,c) - I^{\lim}) V(t+1, D(x,c), I^{\lim}) \leq 1.
\end{aligned}$$

Hence

$$V(t,x,I^{\lim}) = 1 = \sup_{c \in C(x)} \mathbb{1}_{\mathbb{R}^p_+}(I(x,c) - I^{\lim}) V(t+1, D(x,c), I^{\lim}).$$

Now assume $I^{\lim} \notin \mathcal{S}^T(t,x)$ (i.e., $V(t,x,I^{\lim}) = 0$). From (14) in Proposition 3.3 we deduce that for all $c \in C(x)$ one has that $I(c,x) \geq I^{\lim}$ is not true or $I^{\lim} \notin \mathcal{S}^T(t+1, D(x,c))$. That is

$$0 = \mathbb{1}_{\mathbb{R}^p_+}(I(x,c) - I^{\lim})\, \mathbb{1}_{\mathcal{S}^T(t+1,D(x,c))}(I^{\lim}) = \mathbb{1}_{\mathbb{R}^p_+}(I(x,c) - I^{\lim}) V(t+1, D(x,c), I^{\lim})$$

for all $c \in C(x)$, and therefore

$$V(t,x,I^{\lim}) = 0 = \sup_{c \in C(x)} \mathbb{1}_{\mathbb{R}^p_+}(I(x,c) - I^{\lim}) V(t+1, D(x,c), I^{\lim}),$$

which allows to conclude the proof.

Cahiers du GREThA
Working papers of GREThA

GREThA UMR CNRS 5113

Université de Bordeaux

Avenue Léon Duguit
33608 PESSAC - FRANCE
Tel : +33 (0)5.56.84.25.75
Fax : +33 (0)5.56.84.86.47

http://gretha.u-bordeaux.fr/

Cahiers du GREThA (derniers numéros – last issues)

2017-15: MOYES Patrick, EBERT Udo, The Impact of Talents and Preferences on Income Inequality

2017-16: RAZAFIMANDIMBY Andrianjaka Riana, ROUGIER Eric: What difference does it make (when a middle-income country is caught in the trap)? An evidence-based survey analysis of the determinants of Middle-Income Traps

2017-17: LECHEVALIER Sébastien, DEBANES Pauline, SHIN Wonkyu: Financialization and industrial policies in Japan and Korea: Evolving institutional complementarities and loss of state capabilities

2017-18: NAVARRO Noemí, VESZTEG Róbert: On the empirical validity of axioms in unconstrained bargaining

2017-19: LACOUR Claude, GAUSSIER Nathalie: Un écosystème sur la vague ? L'arrivée de la LGV à Bordeaux et l'écosystème start-up

2017-20 : FRIGANT Vincent, MIOLLAN Stéphane, PRESSE Maëlise, VIRAPIN David : Quelles frontières géographiques pour le Technological Innovation System du véhicule à pile à combustible ? Une analyse du portefeuille des co-brevets des constructeurs automobiles

2017-21 : LEVASSEUR Pierre, ORTIZ-HERNANDEZ Luis : Comment l'obésité infantile affecte la réussite scolaire ? Contributions d'une analyse qualitative mise en place à Mexico

2017-22 : A. LAGARDE, A. AHAD-CISSÉ, S. GOURGUET, O. LE PAPE, O. THÉBAUD, N. CAILL-MILLY, G. MORANDEAU, C. MACHER, L. DOYEN: How MMEY mitigates bio-economic impacts of climate change on mixed fisheries

2018-01: J. BALLET, D. POUCHAIN : Fair Trade and the Fetishization of Levinasian Ethics

2018-02: J. VAN DER POL: Explaining the structure of collaboration networks: from firm-level strategies to global network structure

2018-03: L. DOYEN, C. BENE : A generic metric of resilience from resistance to transformation

---

La coordination scientifique des Cahiers du GREThA est assurée par Valerio STERZI.
La mise en page et la diffusion sont assurées par Julie VISSAGUET