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Contests with an uncertain number of prizes

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Abstract

We study multiple-prize contests where the number of prizes to be awarded is a random variable. We determine the symmetric Nash equilibrium of the contest game. We analyze the equilibrium outcome from the perspective of a contest designer aiming at maximizing the aggregate contest expenditure. Assuming that the total value at stake is non-increasing in the number of prizes, we show that the aggregate contest expenditure decreases with the expectation on the number of prizes (first-order stochastic dominance), with the risk in the number of prizes (second-order stochastic dominance), and increases with the number of contestants. We give sufficient conditions such that the same holds under a general specification. Accordingly, a contest designer aiming at maximizing the aggregate contest expenditure should always award a single prize, reveal this information to the contestants and open the contest game to all potential participants.

Keywords: Contest model · Rent-seeking · Multiple-prizes · Number uncertainty · Incomplete information

Compétition avec un nombre incertain de prix

Résumé

Nous étudions un modèle de compétition où le nombre de prix mis en jeu est une variable aléatoire. Nous déterminons l'équilibre de Nash symétrique du modèle. Nous analysons les résultats du point de vue d'un régulateur dont l'objectif est de maximiser l'effort fourni par l'ensemble des participants. Sous l'hypothèse que la valeur totale en jeu ne dépende pas du nombre de prix, nous montrons que l'effort total diminue avec le nombre espéré de prix (dominance stochastique à l'ordre 1), avec le risque sur le nombre de prix (dominance stochastique à l'ordre 2), et augmente avec le nombre de joueurs. Nous donnons des conditions suffisantes pour lesquelles les mêmes résultats tiennent sous une spécification du modèle plus générale. Par conséquent, un régulateur dont l'objectif est de maximiser l'effort total devrait toujours allouer un prix unique, révéler cette information et ouvrir la compétition à tous les participants potentiels.

Mots-clés: Tullock contest · Prix multiples · Incertitude · Information incomplète

JEL: C7, D4, D7, D8

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1 Introduction

Sisak (2009) surveyed the recent literature on multiple-prize contests.[1] She shows that this framework can be relevant in many situations, taking examples from rent-seeking activities, patents and R&D races, licenses, labor markets, sports and so on. Sisak (2009) classifies the literature along two main dimensions, based on the specification of the contest success function (Tullock versus fully discriminating contest success function) and on the adoption of single versus multiple efforts (the contestants exert an overall effort for all prizes or can allocate it more specifically to a sub-group of prizes). The central finding is that with risk-neutral and symmetric contestants, a contest designer aiming at maximizing the aggregate effort should always prefer to allocate a single prize rather than splitting it into several smaller prizes. However, dividing the prize can be optimal in situations with risk aversion (Krishna and Morgan, 1998) and asymmetric players (Szymanski and Valletti, 2005).

Surprizingly, the case of multiple-prize contests with an uncertain number of prizes has never been investigated, although this is a natural assumption and an immediate extension of the literature just surveyed. The purpose of the present paper is to provide a first attempt to fill the gap.

We consider a Tullock contest success function with risk-neutral and symmetric players, assuming that the number of (identical) prizes to be awarded is a random variable, with a common knowledge probability distribution. We characterize the Nash equilibrium of the corresponding contest game and identify some of its properties. Under a specification commonly used in the literature on multiple-prize contests (Berry, 1993; Clark and Riis, 1996), we find that the aggregate effort of the contestants increases when the uncertain number of prizes gets smaller and less risky, in the sense of first-order and second-order stochastic dominance, and when the number of contestants increases. Under the general specification of the model, we give sufficient conditions such that the same properties still prevail.

We also provide a numerical illustration to display less expected behaviors. In line with the literature, our results help to identify situations where a contest designer aiming at maximizing the aggregate effort should always prefer to allocate a single prize with certainty and open the contest game to all potential contestants.

This paper also contributes to the literature from a technical point of view. Our results follow from the properties (monotonicity and concavity) of a *discrete* function first appeared in Clark and Riis (1996). We show here how to extend it to a *continuous and twice differentiable* function, by using of the *psi* (or *digamma*) function (Abramowitz and Stegun, 1964). This greatly eases the analysis of its properties, which are obtained from first-order and second-order derivatives.

[1]The very first contributions on multiple-prize contests are Glazer and Hassin (1988) and Berry (1993).

Finally, this paper is related to Münster (2006), Lim and Matros (2009), Myerson and Wärneryd (2006) and Kahana and Klunover (2015), who extend the contest literature to situations where the number of contestants is uncertain. They show that the (*ex-ante*) aggregate effort in a contest with population uncertainty is smaller than its counterpart in a contest with population certainty and the same expected number of contestants. Clearly, our paper gives the analog finding for contests with prize uncertainty.

The rest of the paper is organized as follows. Section 2 sets out the model. Section 3 characterizes the Nash equilibrium of the contest game. Section 4 deals with the properties of the equilibrium outcome. Section 5 gives the numerical illustration. Section 6 concludes. Several proofs are given in the appendix.

2 The model

We consider n (risk neutral) players competing in a nested contest awarding k prizes, with $1 \leq k < n$. All prizes have the same value, given by the valuation function $V(k)$.[2] The contestants simultaneously exert an overall effort for all prizes in order to get a chance to win one prize and no more. The individual effort of a player i is denoted by x_i and the aggregated effort is denoted by $X = \sum_{j=1}^{n} x_j$. The prizes are awarded in k rounds, according to the iterative process initially described in Clark and Riis (1996). Let $N(\kappa)$ denote the set of players still remaining in the contest at round κ.[3] The conditional probability that a player i in $N(\kappa)$ wins the prize in the κ-th round is equal to

$$p_i^{\kappa}(x_1, ..., x_n) = \begin{cases} \frac{f(x_i)}{\sum_{j \in N(\kappa)} f(x_j)} & \text{if } \sum_{j \in N(\kappa)} f(x_j) > 0 \\ \frac{1}{n+1-\kappa} & \text{if } x_1 = \ldots = x_n \end{cases}$$

where $f(x_i)$, called the impact function (Myerson and Wärneryd, 2006), is a strictly increasing and concave function, with $f(0) = 0$.[4] To ease the notation, it will be useful below to define $g(x_i) \equiv f(x_i)/f'(x_i)$.[5] If player i is drawn at random during round κ, the set of remaining players then evolves according to

$$N(\kappa + 1) = N(\kappa) - \{i\}.$$

The process is repeated until all prizes are allocated. Let $P_i(x_1, ..., x_n; k)$ denote the (*ex ante*) probability that player i wins one prize during this iterative process. Assuming that

[2]For technical reasons, it will be convenient to assume below that $V(k)$ is defined and twice differentiable for all $k \in [1, n]$.

[3]Clearly, $N(1) = \{1, ..., n\}$.

[4]This contest success function is commonly used in the literature and has been axiomatized by Skaperdas (1996).

[5]Remark that $g(x_i)$ is increasing. Indeed, $f(x_i) > 0$ and $f''(x_i) \leq 0$ imply that $g'(x_i) = 1 - f(x_i)f''(x_i)/(f'(x_i))^2 > 0$.

$x_j = x$, for all $j \neq i$, Clark and Riis (1996) show that

$$P_i(x_1, ..., x_n; k) = \frac{f(x_i)}{f(x_i) + (n-1) f(x)} + \sum_{\kappa=2}^{k} \left[\prod_{\lambda=1}^{\kappa-1} \left(1 - \frac{f(x_i)}{f(x_i) + (n-\lambda) f(x)} \right) \right] \frac{f(x_i)}{f(x_i) + (n-\kappa) f(x)}.$$

The originality of our paper is that we assume that the players are unaware of the exact number of prizes to be awarded and only know that this number is distributed between 1 and K, according to a probability distribution $\pi(k)$. In this setting with an uncertain number of prizes, each player i expects to win one prize with probability

$$\mathbb{E}\left[P_i(x_1, ..., x_n; k)\right] = \sum_{k=1}^{K} \pi(k) P_i(x_1, ..., x_n; k).$$

In the literature on multiple-prize contests (Berry, 1993; Clark and Riis, 1996), it is standard to use the specification where $f(x) = x^r$, with $r > 0$, and $V(k) = V/k$. To ease the comparison, below we will sometimes refer to it as a benchmark case. However, it is worth noting that most of the analysis and results will be derived within the general model.

3 Equilibrium outcomes

We characterize the Nash equilibrium of the contest game here. We consider first the case where the players observe the number of prizes k to be allocated before they choose their level of effort. We then deal with the case where the players only know that the number of prizes k is distributed between 1 and K, according to the probability distribution $\pi(k)$.

The case where the number of prizes is known with certainty is solved in Clark and Riis (1996). Observing k, each player i chooses x_i to maximize

$$P_i(x_1, ..., x_n; k) V(k) - x_i.$$

An equilibrium of the contest game satisfies the following first-order condition

$$\frac{\partial}{\partial x_i} P_i(x_1, ..., x_n; k) V(k) - 1 = 0, \text{ for all } i.$$

Clark and Riis (1996) have shown that the contest game admits a symmetric equilibrium. Accordingly, letting $x_i = x$ for all i, we can calculate that[6]

$$\frac{\partial}{\partial x_i} P_i(x, ..., x; k) = \frac{n-k}{n} \left(\sum_{\kappa=1}^{n} \frac{1}{\kappa} - \sum_{\kappa=1}^{n-k} \frac{1}{\kappa} \right) \frac{f'(x)}{f(x)}. \tag{1}$$

[6] For reasons that will become clear below, the expression of the first-order condition used here, though equivalent, differs from that in Clark and Riis (1996). The proof is given in the appendix.

It follows that the game admits a Nash equilibrium such that all players exert an effort satisfying

$$g(x) = A(k, n), \tag{2}$$

where we denote[7]

$$A(k, n) = V(k) \frac{n-k}{n} \left(\sum_{\kappa=1}^{n} \frac{1}{\kappa} - \sum_{\kappa=1}^{n-k} \frac{1}{\kappa} \right). \tag{3}$$

Consider now the case where the players choose their effort under uncertainty. Knowing that k is distributed between 1 and K, according to a probability distribution $\pi(k)$, each player maximizes his expected utility according to his own effort

$$\sum_{k=1}^{K} \pi(k) P_i(x_1, ..., x_n; k) V(k) - x_i.$$

A symmetric equilibrium of the contest game, where $x_i = x$ for all i, satisfies the following first-order condition

$$\sum_{k=1}^{K} \pi(k) \frac{\partial}{\partial x_i} P_i(x, ..., x; k) V(k) - 1 = 0, \text{ for all } i.$$

Using (1), this game admits a symmetric Nash equilibrium where all players exert an effort satisfying

$$g(x) = \sum_{k=1}^{K} \pi(k) A(k, n). \tag{4}$$

4 Comparative statics

Here we derive some comparative statics of the equilibrium outcome under uncertainty. We adopt the point of view of a contest designer aiming at maximizing the aggregate contest expenditure. By assumption, the parameters that the contest designer may be able to manipulate are the probability distribution of the number of prizes and the number of participants.

4.1 Distribution of the number of prizes

We consider here the choice of the probability distribution of the number of prizes. Formally, we compare the equilibrium outcome under two distributions, denoted by $\underline{\pi}(k)$ and $\overline{\pi}(k)$. We let $\underline{x}$ be the equilibrium effort in the contest with $\underline{\pi}(k)$, and $\overline{x}$ the equilibrium effort in the contest with $\overline{\pi}(k)$. We search for conditions on the valuation function $V(k)$[8] such that $\underline{x} > \overline{x}$, considering in turn first-order and second-order stochastic dominance.

[7] Clark and Riis (1996) write $A(k, n) = V(k) \left(k - \sum_{j=0}^{k-1} (k-j) / (n-j) \right) / n$, which is equivalent.

[8] Clark and Riis (1996) consider a general function $V(k)$ such that $V'(k) \leq 0$ before restricting themselves to $V(k) = V/k$.

From our previous analysis, we know that the equilibrium efforts respectively satisfy

$$g\left(\underline{x}\right)=\sum_{k=1}^{K}\underline{\pi}\left(k\right)A\left(k,n\right)\text{ and }g\left(\overline{x}\right)=\sum_{k=1}^{K}\overline{\pi}\left(k\right)A\left(k,n\right).$$

As $g\left(x\right)$ is increasing, the condition for $\underline{x}>\overline{x}$ writes

$$\sum_{k=1}^{K}\underline{\pi}\left(k\right)A\left(k,n\right)>\sum_{k=1}^{K}\overline{\pi}\left(k\right)A\left(k,n\right),\tag{5}$$

where we recall that

$$A\left(k,n\right)=V\left(k\right)\frac{n-k}{n}\left(\sum_{\kappa=1}^{n}\frac{1}{\kappa}-\sum_{\kappa=1}^{n-k}\frac{1}{\kappa}\right).$$

Below, we investigate properties of the valuation function $V\left(k\right)$ such that inequality (5) will hold true, dealing in turn with first-order and second-order stochastic dominance. Assuming that $\overline{\pi}(k)$ first-order stochastically dominates $\underline{\pi}(k)$, it will be sufficient to find shapes of the valuation function $V\left(k\right)$ generating a function $A\left(k,n\right)$ decreasing in k (Courtault et al., 2006). Assuming that $\underline{\pi}(k)$ second-order stochastically dominates $\overline{\pi}(k)$, it will be sufficient to identify shapes of the valuation function $V\left(k\right)$ inducing a function $A\left(k,n\right)$ concave in k (Rothschild and Stiglitz, 1971).

Remarking that $A\left(k,n\right)$ is a discrete function of k[9], our strategy for investigating its properties is as follows. Using the *psi* (or *digamma*) function (Abramowitz and Stegun, 1964), for all *real* numbers $k\in[1,n]$, we first construct a continuous and twice differentiable function $F\left(k\right)$, coinciding with $A\left(k,n\right)$ for every integer k[10]. We then calculate the first-order and second-order derivatives of $F\left(k\right)$. We finally use the derivatives to give sufficient conditions on $V(k)$ ensuring the monotonicity and the concavity of $A\left(k,n\right)$ with respect to k.

For all $k\in[1,n]$, let us define

$$F\left(k\right)=V\left(k\right)\frac{n-k}{n}\left(\psi\left(n+1\right)-\psi\left(n-k+1\right)\right),$$

where ψ is the *psi* (or *digamma*) function (Abramowitz and Stegun, 1964), defined for any positive real χ by

$$\psi\left(\chi\right)=-\gamma+\sum_{j=0}^{\infty}\frac{\chi-1}{\left(j+1\right)\left(j+\chi\right)},\tag{6}$$

where γ is the *Euler constant*. Knowing that

$$\psi\left(\eta+1\right)=-\gamma+\sum_{j=1}^{\eta}\frac{1}{j},$$

[9]This is due to the presence of the term $\sum_{r=1}^{n-k}1/r$.

[10]Formally, $F\left(k\right)=A\left(k,n\right)$ when $k=1,...,n$.

where η denotes any positive integer (Abramowitz and Stegun, 1964), we can verify that

$$F(k) = V(k)\frac{n-k}{n}\left(\sum_{\kappa=1}^{n}\frac{1}{\kappa} - \sum_{\kappa=1}^{n-k}\frac{1}{\kappa}\right) = A(k,n)$$

for all integer values $k = 1, ..., n$. Moreover, knowing that the first-order and second-order derivatives of ψ are (Abramowitz and Stegun, 1964)

$$\psi'(k) = \sum_{j=0}^{\infty}\frac{1}{(j+k)^2} \text{ and } \psi''(k) = -\sum_{j=0}^{\infty}\frac{2}{(j+k)^3}, \tag{7}$$

we know that $F(k)$ is twice differentiable and we can calculate that

$$F'(k) = -\frac{1}{n}\begin{pmatrix} -\left(V'(k)k + V(k)\right)\sum_{j=0}^{\infty}\frac{n-k}{(j+n+1)(j+n-k+1)} \\ +V(k)k\sum_{j=0}^{\infty}\frac{j+1}{(j+n+1)(j+n-k+1)^2} \end{pmatrix} \tag{8}$$

and

$$F''(k) = -\frac{2}{n}\begin{pmatrix} -\frac{1}{2}\left(V''(k)k + 2V'(k)\right)\sum_{j=0}^{\infty}\frac{n-k}{(j+n+1)(j+n-k+1)} \\ +\left(V'(k)k + V(k)\right)\sum_{j=0}^{\infty}\frac{j+1}{(j+n+1)(j+n-k+1)^2} \\ +V(k)k\sum_{j=0}^{\infty}\frac{j+1}{(j+n+1)(j+n-k+1)^3} \end{pmatrix}. \tag{9}$$

Before we derive general results, it is worth considering the benchmark specification, where $V(k) = V/k$, which is the one used in the literature (Berry, 1993; Clark and Riis, 1996). We then obtain clearcut results, as the two conditions respectively simplify to

$$F'(k) = -\frac{V}{n}\sum_{j=0}^{\infty}\frac{j+1}{(j+n+1)(j+n-k+1)^2} < 0$$

and

$$F''(k) = -2\frac{V}{n}\sum_{j=0}^{\infty}\frac{j+1}{(j+n+1)(j+n-k+1)^3} < 0.$$

Accordingly, the equilibrium effort $\underline{x}$ will be larger than $\overline{x}$ whenever $\overline{\pi}(k)$ first-order stochastically dominates $\underline{\pi}(k)$ or $\underline{\pi}(k)$ second-order stochastically dominates $\overline{\pi}(k)$. The first result parallels that of Clark and Riis (1996), showing that when the number of prizes is certain, the contestants expend more in a contest with a single prize than in a contest with several prizes. Indeed, roughly speaking, our result means that when the number of prizes is uncertain, the contestants exert more effort in the contest that awards the smallest expected number of prizes (in the sense of first-order stochastic dominance). The second result is new in the literature. This means that when the number of prizes is uncertain, the contestants expend more in the contest with less risk in the number of prizes (in the sense of second-order stochastic dominance).

Back to the general model, we can extend our results in several directions. Using condition (8), we see that $F'(k) < 0$ whenever the *aggregate* value at stake (i.e., $V(k)k$) is non-increasing in the number of prizes to be allocated (i.e., $V'(k)\,k+V(k)\leq 0$). Then, assuming that $\overline{\pi}(k)$ first-order stochastically dominates $\underline{\pi}(k)$, condition (5) is true and $\underline{x}>\overline{x}$. In other words, the contestants exert more effort in the contest with $\underline{\pi}(k)$ than in the contest with $\overline{\pi}(k)$. Likewise, using (9), we see that $F''(k)<0$ whenever the *aggregate* value at stake (i.e., $V(k)k$) is non-decreasing and weakly concave in the number of prizes to be allocated (i.e., $V'(k)\,k+V(k)\geq 0$ and $V''(k)\,k+2V'(k)\leq 0$). Then, assuming that $\underline{\pi}(k)$ second-order stochastically dominates $\overline{\pi}(k)$, condition (5) is true and $\underline{x}>\overline{x}$, meaning that the contestants expend more in the contest with $\underline{\pi}(k)$ than in the contest with $\overline{\pi}(k)$. Finally, it should also be noted that the set of functions $V(k)$ such that $F'(k)<0$ and $F''(k)<0$, is stable to positive combinations of its elements. Indeed, it should be clear that if $V_1(k)$ and $V_2(k)$ are two valuation functions satisfying $F'(k)<0$ and $F''(k)<0$, so is the valuation function $V(k)=\lambda V_1(k)+\mu V_2(k)$, with $\lambda>0$ and $\mu>0$.

4.2 Number of contestants

We now consider the choice of the number of participants in the contest. Formally, we first derive the comparative statics of the *aggregate* contest expenditure $X=nx$ with respect to the number of contestants n. We then provide conditions on the impact function such that X is increasing in n.

From our previous analysis, we know that the equilibrium effort satisfies

$$g(x)=\sum_{k=1}^{K}\pi(k)\,A(k,n)\,.$$

Using the following expression of $A(k,n)$, found in Clark and Riis (1996),

$$A(k,n)=\frac{V(k)}{n}\left(k-\sum_{j=1}^{k-1}\frac{k-j}{n-j}\right),$$

and letting

$$B(n)\equiv\sum_{k=1}^{K}\pi(k)\,\frac{V(k)}{n}\left(k-\sum_{j=1}^{k-1}\frac{k-j}{n-j}\right),$$

the equilibrium effort is implicitly defined by

$$g(x)=B(n).$$

Using the implicit function theorem, we can show that

$$\frac{\mathrm{d}x}{\mathrm{d}n}=\frac{B'(n)}{g'(x)}.$$

By differentiation of $X=nx$, we then find that

$$\frac{\mathrm{d}X}{\mathrm{d}n}=x+n\frac{B'(n)}{g'(x)}.$$

From this, knowing that $g(x) = B(n)$ and $g'(x) > 0$, we can show that

$$\frac{\mathrm{d}X}{\mathrm{d}n} > 0 \Leftrightarrow \frac{g'(x)\,x}{g(x)} > -\frac{B'(n)\,n}{B(n)}.$$

Now, knowing that

$$B(n)\,n = \sum_{k=1}^{K} \pi(k)\,V(k)\left(k - \sum_{j=1}^{k-1} \frac{k-j}{n-j}\right),$$

we see that $B(n)\,n$ is increasing in n, implying that

$$B(n) + B'(n)\,n > 0$$

and

$$-\frac{B'(n)\,n}{B(n)} < 1.$$

Therefore, a sufficient condition for $dX/dn > 0$ is

$$\frac{g'(x)\,x}{g(x)} \geq 1.$$

In other words, the aggregate expenditure increases with the number of contestants whenever the elasticity of $g(x)$ is at least equal to one. In particular, this condition holds true with the benchmark specification $f(x) = x^r$, for all r.[11]

5 Numerical illustration

In this section, we present a numerical illustration to complete our results. We use the specification where $f(x) = x^r$, with $r = 1$, and $V(k) = V/k^{\mu}$, with $V = 1$ and $\mu \geq 0$. The number of contestants n is set equal to 100. We consider a class of probability distributions $\pi(k)$, characterized by two parameters $1 < \overline{k} < 8$ and $0 < \upsilon < 1$, as represented in Figure 1.[12] Accordingly, the number of prizes has an expected value equal to $\overline{k}$ and a variance equal to υ.

This class of probability distributions is convenient to deal separately with the first- and second-order stochastic dominance effects that we are interested in. An increase in $\overline{k}$ alone gives a new probability distribution that first-order stochastically dominates the initial one. A decrease in υ alone yields a new probability distribution that second-order stochastically dominates the initial one.

Let us first illustrate the effect of a larger expected number of prizes. Figure 2 represents the total effort X as a function of the expected number of prizes $\overline{k}$, the variance υ being set to

[11] Indeed, when $f(x) = x^r$, we have $g(x) = f(x)/f'(x) = x/r$ and $g'(x)\,x/g(x) = 1$.

[12] Formally, $\pi(\overline{k}) = 1 - \upsilon$, $\pi(\overline{k} - 1) = \pi(\overline{k} + 1) = \upsilon/2$ and $\pi(\overline{k}) = 0$ otherwise.

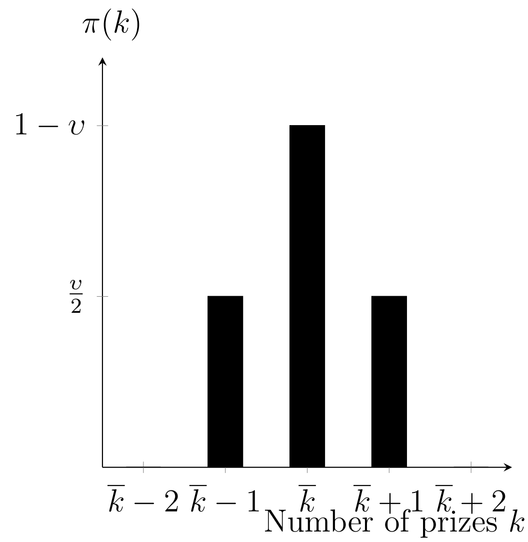
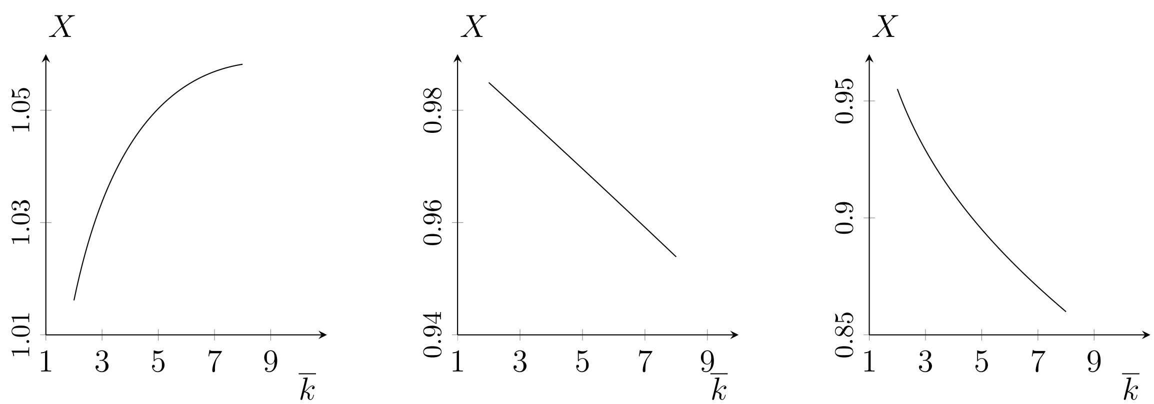
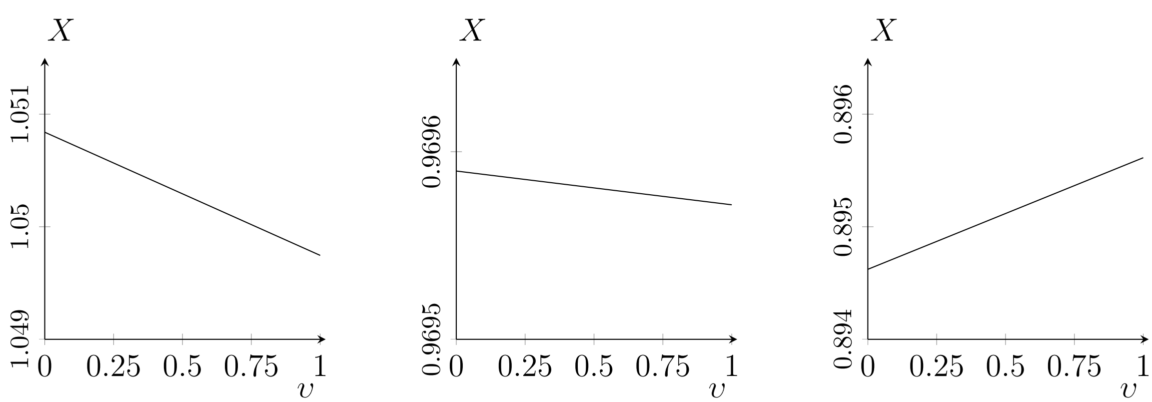


Figure 1: Probability distribution on the number of prizes



(a) $\mu = 0.95$ (b) $\mu = 1$ (c) $\mu = 1.05$

Figure 2: Total effort as a function of $\overline{k}$



(a) $\mu = 0.95$ (b) $\mu = 1$ (c) $\mu = 1.05$

Figure 3: Total effort as a function of υ

0.5. The two panels 2(b) and 2(c), dealing with the cases where $\mu = 1$ and $\mu = 1.05$, are meant to illustrate our findings in Section 4. We observe that the aggregate effort is decreasing as the expected number of prizes increases. In fact, we have shown theoretically that this will be the case whenever $V(k)k$ is non increasing (i.e., here, for all $\mu \geq 1$). Panel 2(a) supplies an example, dealing with the case where $\mu = 0.95$, such that the aggregate effort is increasing with the expected number of prizes $\overline{k}$.

Let us now illustrate the effect of a larger variance of the number of prizes. Figure 3 plots the total effort X as a function of the variance υ, the expected number of prizes $\overline{k}$ being set to 5. The two panels 3(a) and 3(b), dealing with the cases where $\mu = 0.95$ and $\mu = 1$, are given to confirm our results in Section 4. We observe that the aggregate effort is decreasing as the variance of the number of prizes increases. In fact, we have proved theoretically that this will be the case whenever $V(k)k$ is non decreasing and weakly concave (i.e., here, for all $\mu \leq 1$). Panel 3(c) provides an example, dealing with the case where $\mu = 1.05$, such that the aggregate effort is increasing with the variance of the number of prizes υ.

6 Conclusion

This paper is the first one to analyze the case of multiple-prize contests with an uncertain number of prizes. We extend the multiple-prize contest initiated in Berry (1993) and in Clark and Riis (1996), assuming that the number of (identical) prizes to be awarded is a random variable.

Using the same specification as Clark and Riis (1996) (contest success function and prize valuation function), we find that the aggregate contest expenditure increases when the number of prizes to be awarded is expected to be smaller and less risky, in the sense of first-order and second-order stochastic dominance, and when the number of contestants increases. Back to the general model, we check the robustness of these properties, both by providing sufficient conditions such that they still hold true and by proposing numerical illustrations showing opposite results.

This paper also illustrates how to use the psi (or digamma) function (Abramowitz and Stegun, 1964) in order to extend a discrete function appearing in Clark and Riis (1996) to a continuous and twice differentiable function, which helps derive the comparative statics of the equilibrium outcome. Our belief is that this trick may prove to be useful in other settings in the contest literature.

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Appendix

Proof of equation (1)

Clark and Riis (1996) have shown that the contest game admits a symmetric equilibrium. From this, considering player i's point of view, we assume that $x_j = x$, for all $j \neq i$. Then, for all round κ, we can write

$$p_i^{\kappa}\left(x_1, ..., x_n\right) = \frac{f(x_i)}{f(x_i) + (n-\kappa)\, f(x)}.$$

If $k = 1$, player i chooses x_i to maximize

$$\frac{f(x_i)}{f(x_i) + (n-1)\, f(x)} V\left(1\right) - x_i.$$

Therefore, the first-order condition for x_i to maximize player i's payoff is

$$\frac{(n-1)\, f(x) f'(x_i)}{\left(f(x_i) + (n-1)\, f(x)\right)^2} V\left(1\right) - 1 = 0.$$

If $x_i = x$, this simplifies to

$$V\left(1\right) \frac{n-1}{n^2} \frac{f'(x)}{f(x)} - 1 = 0.$$

If $k > 1$, player i chooses x_i to maximize

$$P_i\left(x_1, ..., x_n; k\right) V\left(k\right) - x_i,$$

where

$$\begin{aligned} P_i\left(x_1, ..., x_n; k\right) = & \frac{f(x_i)}{f(x_i) + (n-1)\, f(x)} \\ & + \sum_{\kappa=2}^{k} \left[\prod_{\lambda=1}^{\kappa-1} \left(1 - \frac{f(x_i)}{f(x_i) + (n-\lambda)\, f(x)} \right) \right] \frac{f(x_i)}{f(x_i) + (n-\kappa)\, f(x)} \end{aligned}$$

An equivalent expression is

$$P_i\left(x_1, ..., x_n; k\right) = \sum_{\kappa=1}^{k} \frac{f(x_i)}{(n-\kappa)\, f(x)} \left(\prod_{\lambda=1}^{\kappa} \frac{(n-\lambda)\, f(x)}{f(x_i) + (n-\lambda)\, f(x)} \right).$$

By differentiation, we can get

$$\frac{\partial}{\partial x_i} P_i\left(x_1, ..., x_n; k\right) = \sum_{\kappa=1}^{k} \left[\begin{array}{c} \frac{f'(x_i)}{(n-\kappa) f(x)} \left(\prod_{\lambda=1}^{\kappa} \frac{(n-\lambda) f(x)}{f(x_i)+(n-\lambda) f(x)} \right) \\ + \frac{f(x_i)}{(n-\kappa) f(x)} \left(- \sum_{\lambda=1}^{\kappa} \frac{(n-\lambda) f(x) f'(x_i)}{(f(x_i)+(n-\lambda) f(x))^2} \prod_{\rho \neq \lambda} \frac{(n-\rho) f(x)}{f(x_i)+(n-\rho) f(x)} \right) \end{array} \right].$$

If $x_i = x$, this simplifies to

$$\frac{\partial}{\partial x_i} P_i(x, ..., x; k) = \frac{1}{n} \sum_{\kappa=1}^{k} \left(1 - \sum_{\lambda=1}^{\kappa} \frac{1}{n+1-\lambda}\right) \frac{f'(x)}{f(x)}.$$

We can show by induction that

$$\sum_{\kappa=1}^{k} \left(1 - \sum_{\lambda=1}^{\kappa} \frac{1}{n-\lambda+1}\right) = (n-k)\left(\sum_{\kappa=1}^{n} \frac{1}{\kappa} - \sum_{\kappa=1}^{n-k} \frac{1}{\kappa}\right),$$

which implies that

$$\frac{\partial}{\partial x_i} P_i(x, ...x; k) = \frac{n-k}{n} \left(\sum_{\kappa=1}^{n} \frac{1}{\kappa} - \sum_{\kappa=1}^{n-k} \frac{1}{\kappa}\right) \frac{f'(x)}{f(x)}$$

Second-order condition A sufficient condition for a global maximum of player i's payoff is obtained if

$$\frac{\partial}{\partial x_i} P_i(x_i, x, ..., x; k) = \sum_{\kappa=1}^{k} \left[\left(\frac{f'(x_i)}{(n-\kappa) f(x)}\right)\left(\prod_{\lambda=1}^{\kappa} \frac{(n-\lambda) f(x)}{f(x_i) + (n-\lambda) f(x)}\right)\left(1 - \sum_{\lambda=1}^{\kappa} \frac{f(x_i)}{f(x_i) + (n-\lambda) f(x)}\right)\right]$$

is decreasing with x_i. If we consider the following condition

$$1 - \sum_{\lambda=1}^{\kappa} \frac{f(x_i)}{f(x_i) + (n-\lambda) f(x)} > 0, \text{ for all } 1 \leq \kappa \leq k,$$

then, the expression under square brackets above is the product of three terms, each positive and decreasing in x_i (given that $f'(x_i) > 0$ and $f''(x_i) \leq 0$). Clearly, this implies that

$$\frac{\partial^2}{\partial (x_i)^2} P_i(x_i, x, ..., x; k) < 0,$$

which gives a sufficient second-order condition for a global maximum [13].

Nevertheless the proposed inequality is not necessarily satisfied for all x_i[14]. However no player will invest more than $V(k)$ in the contest (otherwise his utility is negative) so we need to verify this condition for all $x_i \in [0; V(k)]$.

Moreover remark that our condition holds true for all $1 \leq \kappa \leq k$ if it holds true for $\kappa = k$. Thus we obtain a sufficient condition for a global maximum:

$$1 - \sum_{\lambda=1}^{k} \frac{f(x_i)}{f(x_i) + (n-\lambda) f(x)} > 0, \text{ for all } x_i \in [0; V(k)], \text{ for all } k.$$

[13](Clark and Riis, 1998) provide a sufficient second-order condition for a local maximum only. It writes $1 - \sum_{\lambda=1}^{\kappa} \frac{1}{n+1-\lambda} > 0$, for all $1 \leq \kappa \leq k$. Clearly, it can be derived from our analysis in the case where $x_i = x$.

[14]For instance, if $x_i \to +\infty$, then $\frac{f(x_i)}{f(x_i)+(n-\lambda)f(x)}$ tends to 1 and the left hand side of the inequality tends to $1 - \kappa \leq 0$.

Given that this condition may not be necessarily fulfilled, we can at least show that player i's utility function is concave over the interval $[0; x]$ if the number of prizes is such that

$$\sum_{\lambda=1}^{k} \frac{1}{n-\lambda+1} \leq 1,$$

which can be rewritten as

$$\sum_{\kappa=1}^{n} \frac{1}{\kappa} - \sum_{\kappa=1}^{n-k} \frac{1}{\kappa} \leq 1,$$

and approximated by $k \leq 0.632n$ for large values of n according to Clark and Riis (1998). Indeed, since $f(x_i) = 0$ when $x_i = 0$ and given that

$$\sum_{\lambda=1}^{k} \frac{f(x_i)}{f(x_i) + (n-\lambda) f(x)}$$

is increasing with x_i, our condition

$$1 - \sum_{\lambda=1}^{k} \frac{f(x_i)}{f(x_i) + (n-\lambda)\, f(x)} > 0$$

is true for all $x_i \in [0; x]$.

Proof of equations (8) and (9)

The function F is twice differentiable; we can calculate $F'(k)$ and $F''(k)$:

$$\begin{aligned} F'(k) = {} & \frac{V'(k)}{n} (n-k) \left(\psi(n+1) - \psi(n-k+1) \right) \\ & - \frac{V(k)}{n} \left(\psi(n+1) - \psi(n-k+1) - (n-k)\psi'(n-k+1) \right) \end{aligned}$$

$$\begin{aligned} F''(k) = {} & \frac{V''(k)}{n} (n-k) \left(\psi(n+1) - \psi(n-k+1) \right) \\ & - 2\frac{V'(k)}{n} \left(\psi(n+1) - \psi(n-k+1) - (n-k)\, \psi'(n-k+1) \right) \\ & - \frac{V(k)}{n} \left(2\psi'(n-k+1) + (n-k)\psi''(n-k+1) \right). \end{aligned}$$

We report some intermediate calculations using (7)

$$\begin{aligned} \psi(n+1) - \psi(n-k+1) &= \sum_{j=0}^{\infty} \frac{k}{(j+n+1)(j+n-k+1)} \\ \psi'(n-k+1) &= \sum_{j=0}^{\infty} \frac{1}{(j+n-k+1)^2} \\ \psi''(n-k+1) &= -\sum_{j=0}^{\infty} \frac{2}{(j+n-k+1)^3} \end{aligned}$$

which allow us to rewrite $F'(k)$ and $F''(k)$ as

$$F'(k) = \frac{V'(k)}{n} \sum_{j=0}^{\infty} \frac{(n-k)\,k}{(j+n+1)(j+n-k+1)} - \frac{V(k)}{n} \left(\sum_{j=0}^{\infty} \frac{k}{(j+n+1)(j+n-k+1)} - \sum_{j=0}^{\infty} \frac{n-k}{(j+n-k+1)^2} \right)$$

$$\begin{aligned} F''(k) = {} & \frac{V''(k)}{n} \sum_{j=0}^{\infty} \frac{(n-k)\,k}{(j+n+1)(j+n-k+1)} \\ & - 2\frac{V'(k)}{n} \left(\sum_{j=0}^{\infty} \frac{k}{(j+n+1)(j+n-k+1)} - \sum_{j=0}^{\infty} \frac{n-k}{(j+n-k+1)^2} \right) \\ & - 2\frac{V(k)}{n} \left(\sum_{j=0}^{\infty} \frac{1}{(j+n-k+1)^2} - \sum_{j=0}^{\infty} \frac{n-k}{(j+n-k+1)^3} \right). \end{aligned}$$

First we focus on $F'(k)$:

$$F'(k) = -\frac{1}{n} \begin{pmatrix} -V'(k)\,k \sum_{j=0}^{\infty} \frac{n-k}{(j+n+1)(j+n-k+1)} \\ +V(k) \left(\sum_{j=0}^{\infty} \frac{k}{(j+n+1)(j+n-k+1)} - \sum_{j=0}^{\infty} \frac{n-k}{(j+n-k+1)^2} \right) \end{pmatrix}$$

We add and remove the following expression

$$V(k) \sum_{j=0}^{\infty} \frac{n-k}{(j+n+1)(j+n-k+1)},$$

which leads to

$$\begin{aligned} F'(k) &= -\frac{1}{n} \begin{pmatrix} -(V'(k)\,k + V(k)) \sum_{j=0}^{\infty} \frac{n-k}{(j+n+1)(j+n-k+1)} \\ +V(k) \left(\sum_{j=0}^{\infty} \frac{k}{(j+n+1)(j+n-k+1)} - \sum_{j=0}^{\infty} \frac{n-k}{(j+n-k+1)^2} + \sum_{j=0}^{\infty} \frac{n-k}{(j+n+1)(j+n-k+1)} \right) \end{pmatrix} \\ &= -\frac{1}{n} \begin{pmatrix} -(V'(k)\,k + V(k)) \sum_{j=0}^{\infty} \frac{n-k}{(j+n+1)(j+n-k+1)} \\ +V(k) \left(\sum_{j=0}^{\infty} \frac{n}{(j+n+1)(j+n-k+1)} - \sum_{j=0}^{\infty} \frac{n-k}{(j+n-k+1)^2} \right) \end{pmatrix} \end{aligned}$$

We finally obtain expression (8):

$$F'(k) = -\frac{1}{n} \begin{pmatrix} -(V'(k)\,k + V(k)) \sum_{j=0}^{\infty} \frac{n-k}{(j+n+1)(j+n-k+1)} \\ +kV(k) \sum_{j=0}^{\infty} \frac{j+1}{(j+n+1)(j+n-k+1)^2} \end{pmatrix}$$

Looking at $F''(k)$:

$$F''(k) = -\frac{2}{n} \begin{pmatrix} -\frac{1}{2} V''(k)\,k \sum_{j=0}^{\infty} \frac{n-k}{(j+n+1)(j+n-k+1)} \\ +V'(k) \left(\sum_{j=0}^{\infty} \frac{k}{(j+n+1)(j+n-k+1)} - \sum_{j=0}^{\infty} \frac{n-k}{(j+n-k+1)^2} \right) \\ +V(k) \left(\sum_{j=0}^{\infty} \frac{1}{(j+n-k+1)^2} - \sum_{j=0}^{\infty} \frac{n-k}{(j+n-k+1)^3} \right) \end{pmatrix}$$

We add and remove the following expression in the first two terms of the sum

$$V'(k)\sum_{j=0}^{\infty}\frac{n-k}{(j+n+1)\,(j+n-k+1)},$$

which leads to

$$F''\left(k\right)=-\frac{2}{n}\left(\begin{array}{c}-\frac{1}{2}(V''\left(k\right)k+2V'(k))\sum_{j=0}^{\infty}\frac{n-k}{(j+n+1)(j+n-k+1)}\\+V'\left(k\right)\left(\sum_{j=0}^{\infty}\frac{k}{(j+n+1)(j+n-k+1)}-\sum_{j=0}^{\infty}\frac{n-k}{(j+n-k+1)^2}+\sum_{j=0}^{\infty}\frac{n-k}{(j+n+1)(j+n-k+1)}\right)\\+V(k)\left(\sum_{j=0}^{\infty}\frac{1}{(j+n-k+1)^2}-\sum_{j=0}^{\infty}\frac{n-k}{(j+n-k+1)^3}\right)\end{array}\right)$$

Rearranging the terms, it is possible to rewrite $F''(k)$ as

$$F''\left(k\right)=-\frac{2}{n}\left(\begin{array}{c}-\frac{1}{2}(V''\left(k\right)k+2V'(k))\sum_{j=0}^{\infty}\frac{n-k}{(j+n+1)(j+n-k+1)}\\+V'\left(k\right)\left(-\sum_{j=0}^{\infty}\frac{n-k}{(j+n-k+1)^2}+\sum_{j=0}^{\infty}\frac{n}{(j+n+1)(j+n-k+1)}\right)\\+V(k)\sum_{j=0}^{\infty}\frac{j+1}{(j+n-k+1)^3}\end{array}\right).$$

We finally obtain expression (9)

$$F''\left(k\right)=-\frac{2}{n}\left(\begin{array}{c}-\frac{1}{2}\left(V''\left(k\right)k+2V'\left(k\right)\right)\sum_{j=0}^{\infty}\frac{n-k}{(j+n+1)(j+n-k+1)}\\+\left(V'\left(k\right)k+V(k)\right)\sum_{j=0}^{\infty}\frac{j+1}{(j+n+1)(j+n-k+1)^2}\\+V(k)k\sum_{j=0}^{\infty}\frac{j+1}{(j+n+1)(j+n-k+1)^3}\end{array}\right).$$

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